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Some long-range dependence processes arising from fluctuations of particle systems*

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Abstract. Several long-range dependence, self-similar Gaussian processes arise from asymptotics of some classes of spatially distributed particle systems and superprocesses. The simplest examples are fractional Brownian motion and sub-fractional fractional Brownian motion, the latter being intermediate between Brownian motion and fractional Brownian motion. In this paper we focus mainly on long-range dependence processes that arise from occupation time fluctuations of immigration particle systems with or without branching, and we study their properties. Some long-range dependence non-Gaussian processes that appear in a similar way are also mentioned.

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1. Introduction

Long-range dependence processes (also called long-memory processes) and their statistics have many areas of application: finance, econometrics, hydrology, meteorology, turbulence, geophysics, statistical physics, communication networks, neuroscience, analysis of DNA sequences. The literature on the subject is vast and varied; we give only a sample of recent references in applied fields: [8, 9, 10, 11, 12, 15, 20, 22, 28, 31, 33, 36, 37, 38, 41, 44, 46, 47, 49, 50, 54, 57];

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the book [16] surveys long-range dependence from its origins to several recent applications. The phenomenon of long-range dependence may occur in the temporal structure of a process and in the spatial structure of a random field, and the processes and fields may be Gaussian or non-Gaussian, stationary or non-stationary (see e.g. [15, 49] and references therein for long-range dependence random fields).

It is worthwhile to study long-range dependence processes that arise from specific stochastic systems which are also of interest by themselves. In this paper we study long-range dependence, self-similar Gaussian processes related to some spatially distributed particle systems. We focus mainly on occupation time fluctuations of branching and non-branching systems with immigration. Our purpose is to show how various types of long-range dependence processes can be obtained from fluctuations of particle systems. The ways in which such processes arise exhibit interesting properties of the systems and may lead to possible applications.

Long-range dependence may be conceptualized in various ways (see e.g. [29, 30, 45, 54]). What we regard here as long-range dependence for Gaussian processes is the decay of the covariance of increments on intervals according to a negative power of the distance between the intervals. This can be viewed as a form of mixing. All the long-range processes we shall see here are not stationary, and all except fractional Brownian motion do not have stationary increments. (For self-similar processes with stationary increments, the term "long-range dependence" is used for the case where the covariance of the increment process decays so slowly that it is not summable; for fractional Brownian motion this corresponds to Hurst parameter $> 1/2$; see [45, 54]).

The best known and most widely used long-range dependence, self-similar Gaussian process is fractional Brownian motion (fBm), $\xi_h = \{\xi_h(t), t \geq 0\}, h \in (0, 2)$, whose covariance function is

$$C_{\xi_h}(s,t) = \frac{1}{2}[s^h + t^h - (t-s)^h], \quad 0 \leq s \leq t. \tag{1.1}$$

$H = h/2$ is called Hurst parameter (see e.g. [45, 54]). The process ξ_h coincides with Brownian motion (Bm) for $h = 1$, and for $h \neq 1$ it is a limit in distribution of several types of processes (e.g., [1, 10, 12, 13, 18, 36, 43, 46, 51, 53, 56]). Furthermore, ξ_h with $h \in (1, 3/2]$ is also obtained from a scaling limit of occupation time fluctuations of a Poisson system of particles moving independently in $\mathbb{R}^d$ according to a symmetric α-stable Lévy process, $\alpha \in (0, 2]$, under the condition $1 = d < \alpha$, with h given by $h = 2 - 1/\alpha$ [4] (the particle system is described in Section 2). The case $\alpha = 2$ (corresponding to $h = 3/2$) also follows from an occupation time fluctuation limit theorem in [14] (no relation to fBm and long-range dependence is mentioned in that paper).

Another long-range dependence, self-similar Gaussian process, $\zeta_h = \{\zeta_h(t), t \geq 0\}, h \in (0, 2)$, with covariance function

$$C_{\zeta_h}(s,t) = s^h + t^h - \frac{1}{2}[(s+t)^h + (t-s)^h], \quad 0 \leq s \leq t, \tag{1.2}$$

was obtained in [4], and it also coincides with Bm for $h = 1$. The process ζ_h has some of the main properties of ξ_h for $h \neq 1$, in particular it is not a Markov process and not a semimartingale, but it does not have stationary increments and it is more weakly correlated than ξ_h. The main difference between ξ_h and ζ_h regarding long-range dependence is that the covariance of increments on intervals separated by distance τ decays at rate τ^{h-2} for ξ_h and at rate τ^{h-3} for ζ_h as $\tau \to \infty$. Thus the long-range dependence decays faster for ζ_h than for ξ_h. Hence ζ_h is intermediate between Bm and fBm, and that is why it was called "sub-fractional Brownian motion" (sub-fBm). See [4] for a study of properties sub-fBm and comparisons with fBm. A process like sub-fBm was introduced in [17] in a different context.

It is shown in [4] that ζ_h with $h \in (1,2)$ is obtained from a scaling limit of occupation time fluctuations of the above-mentioned particle system where the particles additionally undergo critical binary branching (see the description in Section 2), under the condition $\alpha < d < 2\alpha$, with h given by $h = 3 - d/\alpha$. The case $d = 3, \alpha = 2$ (corresponding to $h = 3/2$) also coincides with an occupation time fluctuation limit in the context of superprocesses stated in [34] without proof (no mention of long-range dependence); see also [32].

In this paper we study the effect on the occupation time fluctuation limits of incorporating immigration in the particles systems, with and without branching. This leads to two new long-range dependence, self-similar Gaussian processes whose covariance functions are given by

$$s^{h+1} + t^{h+1} - (t-s)^h(hs+t), \quad 0 \le s \le t, \quad h \in (1, 3/2], \tag{1.3}$$

without branching, and

$$s^{h+1} + t^{h+1} - \frac{1}{4}(s+t)^{h+1} - \frac{1}{4}(t-s)^h[(2h-1)s+3t], \quad 0 \le s \le t, \quad h \in (1,2), \tag{1.4}$$

with branching.

Another long-range dependence, self-similar Gaussian process with covariance function

$$(s+t)^h - (t-s)^h - 2hs(t-s)^{h-1}, \quad 0 \le s \le t, \quad h \in (1, 3/2], \tag{1.5}$$

arises from fluctuations of the immigration branching system itself (not its occupation time). The special case of this process with $d = 1$, $\alpha = 2$ ($h = 3/2$) is also related to a limit theorem in [39] in the context of superprocesses (no mention of long-range dependence).

Other long-range dependence, self-similar Gaussian processes can be derived from asymptotics of immigration superprocesses, e.g., the super-Brownian motion with super-Brownian immigration studied in [58]; an example is a process with covariance function

$$\frac{2}{3}\left[\left(\frac{s+t}{2}\right)^{3/2} - \left(\frac{t-s}{2}\right)^{3/2}\right] + \frac{1}{\sqrt{2}}(t+s)^2[(t+s)^{-1/2} - (t-s)^{-1/2}], \; 0 \le s \le t. \tag{1.6}$$

Existence of the Gaussian processes with the previous covariances can be established simply by taking limits of the covariance functions of the systems, as we shall see. Although convergence of the covariance alone yields only partial (and not always reliable) information, the results suggest that the fluctuations of the corresponding spatially distributed particle processes should be asymptotically Gaussian with long-range dependence (under suitable conditions on d and α). In order to show that this is so, it is necessary to prove a much stronger convergence, namely, functional convergence of the particle processes. This is done in [5] for the occupation time fluctuations of the particle systems with and without branching, but no immigration, which are related to the covariances (1.1) and (1.2). The proofs involve a higher level of technical difficulty. In this paper we concentrate on existence of the Gaussian processes with the covariances (1.3) and (1.4), and we study their properties. Corresponding functional convergence results may be proved with the methods of [6]. The cases of covariances (1.5) and (1.6) and others can be studied similarly. In all the cases of space-time Gaussian limits we shall see, the covariance of the limit is a product of a spatial covariance and a temporal covariance; the covariances (1.1)-(1.6) are the temporal parts.

In Section 2 we prove existence of the long-range dependence Gaussian processes with covariances (1.3)-(1.5) by showing how they arise from immigration particle systems. As expected when long-range dependence is involved, the normings required for the scaling limits are atypical (i.e., different from the one for the classical central limit theorem). Continuing with particle systems, in Section 3 we show that a non-Markov process called "demographic variation process", which measures the difference between the branching particle system and the system without branching (both without immigration) leads to sub-fBm. Section 4 is devoted to properties of the Gaussian processes with covariances (1.3) and (1.4); this section can be read independently of the others. In Section 5 we make some comments, including remarks on functional limits for the particle systems and on the causes of long-range dependence in the systems, and we mention some long-range dependence non-Gaussian processes that also arise from branching particle systems.

2. Fluctuation of some particle systems

The following theorem gives existence of the Gaussian processes with covariances (1.3) and (1.4). We will prove it from long-time asymptotics of occupation time fluctuations of the particle systems described below.

Theorem 2.1.
(1) There exists a centered Gaussian process $\eta_h = \{\eta_h(t), t \geq 0\}$ *for each* $h \in (1, 3/2]$ *with covariance function*

$$C_{\eta_h}(s,t) = E(\eta_h(s)\eta_h(t)) = s^{h+1} + t^{h+1} - (t-s)^h(hs+t), \quad 0 \leq s \leq t. \tag{2.1}$$

(2) There exists a centered Gaussian process $\sigma_h = \{\sigma_h(t), t \geq 0\}$ *for each* $h \in (1, 2)$ *with covariance function*

$$\begin{aligned} C_{\sigma_h}(s,t) &= E(\sigma_h(s)\sigma_h(t)) \\ &= s^{h+1} + t^{h+1} - \frac{1}{4}(s+t)^{h+1} - \frac{1}{4}(t-s)^h[(2h-1)s+3t], \quad 0 \leq s \leq t, \end{aligned} \tag{2.2}$$

The particle systems on $\mathbb{R}^d$ are described as follows, adding immigration in the ones studied in [4]. At the initial time $t = 0$ particles are distributed according to a Poisson random field on $\mathbb{R}^d$ with Lebesgue intensity measure. Particles immigrate according to a space-time Poisson random field on $\mathbb{R}^d \times \mathbb{R}_+$ with Lebesgue intensity measure. All particles evolve independently, moving according to a symmetric α-stable Lévy process, $\alpha \in (0, 2]$ ($\alpha = 2$ corresponds to Brownian motion), and branching at rate V according to a critical binary branching law (i.e., at an exponentially distributed lifetime with parameter V a particle dies or splits into two particles, each case with probability 1/2). Note that putting $V = 0$ we have the system without branching. Let $N = \{N_t, t \geq 0\}$ denote the measure-valued process such that $N_t(A)$ is the number of particles in a set $A \subset \mathbb{R}^d$ at time t. This is clearly a Markov process.

We use the following notation: $p_t(x, y) = p_t(x - y)$ denotes the transition probability density of the α-stable process, $\{\mathcal{T}_t, t \geq 0\}$ denotes its semigroup (i.e, $\mathcal{T}_t\varphi(x) = \int p_t(x, y)\varphi(y)dy$), we take functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (the usual space of smooth rapidly decreasing functions), and we write $\langle \mu, f \rangle = \int f d\mu$, where μ is a measure and f is a function on $\mathbb{R}^d$. Lebesgue measure on $\mathbb{R}^d$ is written λ. Recall that the Fourier transform of φ is defined by $\widehat{\varphi}(z) = \int e^{ix\cdot z}\varphi(x)dx$ where $x \cdot z$ is the scalar product in $\mathbb{R}^d$.

We will simplify the proofs slightly by assuming that there are no initial particles (at time 0). However, we shall see that the results below are the same if the initial particles are included

(Remark 2.4.(a)). With this assumption the mean and the covariance of the process N are given by

$$E\langle N_t, \varphi\rangle = t\langle\lambda, \varphi\rangle, \quad t \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{2.3}$$

and

$$\begin{aligned} &Cov(\langle N_s, \varphi\rangle, \langle N_t, \psi\rangle) \\ &= \; s\langle\lambda, \varphi\mathcal{T}_{t-s}\psi\rangle + V\int_0^s\int_0^{s-r}\langle\lambda, \varphi\mathcal{T}_{t+s-2(r+u)}\psi\rangle du dr, \;\; 0 \leq s \leq t, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \tag{2.4}$$

Formulas of this type in a more general setting are derived by martingale methods based on the Markov property of the process N (see e.g. [26]), and formulas (2.3) and (2.4) are obtained by specializing to the present case (as a consequence of the invariance and the self-adjointness of $\mathcal{T}_t$ with respect to λ, and the critical binary branching).

The rescaled occupation time process $L_T = \{L_T(t), t \geq 0\}$ of N is defined by

$$\langle L_T(t), \varphi\rangle = \int_0^{Tt}\langle N_s, \varphi\rangle ds = T\int_0^t \langle N_{Ts}, \varphi\rangle ds, \;\; t \geq 0, \; \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{2.5}$$

where $T > 0$ is the scaling parameter that will tend to ∞. This process is not Markov.

The fluctuation process $X_T = \{X_T(t), t \geq 0\}$ of L_T is defined by

$$\begin{aligned} \langle X_T(t), \varphi\rangle &= \frac{1}{F_T}(\langle L_T(t), \varphi\rangle - E\langle L_T(t), \varphi\rangle) \\ &= \frac{T}{F_T}\int_0^t(\langle N_{Ts}, \varphi\rangle - E\langle N_{Ts}, \varphi\rangle)ds, \quad t \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \end{aligned} \tag{2.6}$$

where $F_T > 0$ is a norming. From the perspective of particle systems, the aim is to find F_T such that X_T converges in distribution as $T \to \infty$ and to identify the limit process (see Section 5 for comments on functional convergence and space-time Gaussian limits). For our purpose here it suffices to show convergence of the covariance of X_T.

Taking covariance of the process in (2.6) we have

$$\begin{aligned} &Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) = \frac{T^2}{F_T^2}\int_0^s du\int_0^t dv \;\; Cov(\langle N_{Tu}, \varphi\rangle, \langle N_{Tv}, \psi\rangle) \\ &= \frac{T^2}{F_T^2}\left(\int_0^s du\int_s^t dv + \int_0^s du\int_u^s dv + \int_0^s dv\int_v^s du\right)Cov(\langle N_{Tu}, \varphi\rangle, \langle N_{Tv}, \psi\rangle), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad s \leq t, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d). \end{aligned} \tag{2.7}$$

Using the Plancherel formula for Fourier transform and $\widehat{\mathcal{T}_t\varphi}(z) = e^{-t|z|^\alpha}\widehat{\varphi}(z)$ in (2.4), we rewrite (2.7) as

$$\begin{aligned} &Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\ &= \frac{1}{(2\pi)^d}\frac{T^2}{F_T^2}\int_{\mathbb{R}^d}\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}\left(\int_0^s du\int_s^t dv + 2\int_0^s du\int_u^s dv\right) \\ &\quad \left[e^{-T(v-u)|z|^\alpha}Tu + V\int_0^{Tu}\int_0^{Tu-r}e^{-[T(v+u)-2(r+w)]|z|^\alpha}dw dr\right]dz, \quad s \leq t. \end{aligned} \tag{2.8}$$

We want to take the limit of the covariance (2.8) as $T \to \infty$ without the branching ($V = 0$) and with the branching ($V > 0$). The results are given in the next two theorems.

Theorem 2.2. *(No branching, $V = 0$)*

(1) For $(1 =)\, d < \alpha$, with $F_T = T^{(3-d/\alpha)/2}$, and for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
&\lim_{T\to\infty} Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\
&= \langle \lambda, \varphi\rangle\langle \lambda, \psi\rangle \frac{\Gamma(2-h)}{\pi\alpha(h-1)h(h+1)}[s^{h+1} + t^{h+1} - (t-s)^h(hs+t)], \quad s \le t, \qquad (2.9)
\end{aligned}$$

where $h = 2 - d/\alpha$, $h \in (1, 3/2]$.

(2) For $d > \alpha$, with $F_T = T$, and for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{T\to\infty} Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) = \langle \lambda, \varphi G\psi\rangle s^2, \quad s \le t, \qquad (2.10)$$

where G is the potential operator of the symmetric α-stable process in $\mathbb{R}^d$,

$$G\psi(x) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}\int_{\mathbb{R}^d} \frac{\psi(y)}{|x-y|^{d-\alpha}}dy, \;\; d > \alpha, \qquad (2.11)$$

Theorem 2.3. *(Branching, $V > 0$)*
For $\alpha < d < 2\alpha$, with $F_T = T^{(4-d/\alpha)/2}$, and for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}
&\lim_{T\to\infty} Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\
&= \langle \lambda, \varphi\rangle\langle \lambda, \psi\rangle \frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)(h-1)h(h+1)} \\
&\quad \times\Big(s^{h+1} + t^{h+1} - \frac{1}{4}(s+t)^{h+1} - \frac{1}{4}(t-s)^h[(2h-1)s + 3t]\Big), \quad s \le t, \qquad (2.12)
\end{aligned}$$

where $h = 3 - d/\alpha$, $h \in (1, 2)$.

Proof of Theorem 2.2:
(1) Making the change of variable $z = y(T) = (T(v-u))^{-1/\alpha}y$ in (2.8) (with $V = 0$) we have

$$\begin{aligned}
&Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\
&= \frac{1}{(2\pi)^d}\frac{T^{3-d/\alpha}}{F_T^2}\Big(\int_0^s du\int_0^t dv + 2\int_0^s du\int_u^s dv\Big)u(v-u)^{-d/\alpha} \\
&\quad \times \int_{\mathbb{R}^d} \widehat{\varphi}(y(T))\overline{\widehat{\psi}(y(T))}e^{-|y|^\alpha}dy. \qquad (2.13)
\end{aligned}$$

In order to obtain a limit in (2.13) we put $F_T = T^{(3-d/\alpha)/2}$ and we have

$$\begin{aligned}
&\lim_{T\to\infty} Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\
&= \frac{1}{(2\pi)^d}\widehat{\varphi}(0)\overline{\widehat{\psi}(0)}\int_{\mathbb{R}^d} e^{-|y|^\alpha}dy\Big(\int_0^s du\int_s^t dv + 2\int_0^s du\int_u^s dv\Big)u(v-u)^{-d/\alpha}, \quad (2.14)
\end{aligned}$$

but $\widehat{\varphi}(0) = \langle \lambda, \varphi \rangle$, $\overline{\widehat{\psi}(0)} = \langle \lambda, \psi \rangle$,

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-|y|^\alpha} dy = \frac{\Gamma(d/\alpha)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)} = \frac{\Gamma(1/\alpha)}{\pi\alpha},$$

and

$$\begin{aligned}
&\left(\int_0^s du \int_s^t dv + 2 \int_0^s du \int_u^s dv \right) u(v-u)^{-d/\alpha} \\
= \quad & \frac{1}{(1-d/\alpha)(2-d/\alpha)(3-d/\alpha)} \\
& \times \left[s^{3-d/\alpha} + t^{3-d/\alpha} - \left(3 - \frac{d}{\alpha}\right) s(t-s)^{2-d/\alpha} - (t-s)^{3-d/\alpha} \right], \quad d < \alpha.
\end{aligned}$$

Hence, denoting $h = 2 - d/\alpha$ we obtain (2.9) from (2.14).

(2) We have from (2.8) (with $V = 0$), integrating on v,

$$\begin{aligned}
& Cov(\langle X_T(x), \varphi \rangle, \langle X_T(t), \psi \rangle) \\
= \quad & \frac{1}{(2\pi)^d} \frac{T^2}{F_T^2} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{|z|^\alpha} \left[\int_0^s (e^{-T(s-u)|z|^\alpha} - e^{-T(t-u)|z|^\alpha}) u du \right. \\
& \left. +2 \int_0^s (1 - e^{-T(s-u)|z|^\alpha}) u du \right] dz. \qquad (2.15)
\end{aligned}$$

For convergence in (2.15) we put $F_T = T$ and we have

$$\lim_{T\to\infty} Cov(\langle X_T(s), \varphi \rangle, \langle X_T(t), \psi \rangle) = \frac{s^2}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{|z|^\alpha} dz.$$

But

$$\frac{1}{(2\pi)^d} \int \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{|z|^\alpha} dz = \langle \lambda, \varphi G \psi \rangle$$

with G given by (2.11). Hence we obtain (2.10). □

Proof of Theorem 2.3: We write (2.8) as

$$Cov(\langle X_T(s), \varphi \rangle, \langle X_T(t), \psi \rangle) = A_T(s,t) + B_T(s,t), \qquad (2.16)$$

where

$$A_T(s,t) = \frac{1}{(2\pi)^d} \frac{T^2}{F_T^2} \int_{\mathbb{R}^d} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)} \left(\int_0^s du \int_s^t dv + 2 \int_0^s du \int_u^s dv \right) e^{-T(v-u)|z|^\alpha} T u dz$$

and

$$\begin{aligned}
B_T(s,t) \quad = \quad & \frac{V}{(2\pi)^d} \frac{T^2}{F_T^2} \int_{\mathbb{R}^d} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)} \left(\int_0^s du \int_s^t dv + 2 \int_0^s du \int_u^s dv \right) \\
& \left(\int_0^{Tu} \int_0^{Tu-r} e^{-[T(u+v)-2(r+w)]|z|^\alpha} dw dr \right) dz. \qquad (2.17)
\end{aligned}$$

We know from Theorem 2.2.(2) that $A_T(s,t)$ converges with $F_T = T$ as $T \to \infty$ for $d > \alpha$, and on the other hand we shall see that $B_T(s,t)$ converges with $F_T = T^{(4-d/\alpha)/2}$ for $\alpha < d < 2\alpha$; therefore $A_T(s,t)$ with the latter norming goes to zero as $T \to \infty$.

Computing in (2.17) we find

$$\begin{aligned} B_T(s,t) &= \frac{V}{(2\pi)^d} \frac{T^2}{F_T^2} \int_{\mathbb{R}^d} \widehat{\varphi}(z)\overline{\widehat{\psi}(z)} \Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big) \\ & \Big(\frac{Tu}{2|z|^\alpha} e^{-T(v-u)|z|^\alpha} - \frac{T}{4|z|^\alpha} \int_{v-u}^{v+u} e^{-Tr|z|^\alpha} dr \Big) dz. \end{aligned}$$

Making the changes of variables $z = y_1(T) = (T(v-u))^{-1/\alpha} y$ and $z = y_2(T) = (Tr)^{-1/\alpha} y$ in the first and second terms, respectively, we have

$$\begin{aligned} B_T(s,t) &= \frac{V}{4(2\pi)^2} \frac{T^{4-d/\alpha}}{F_T^2} \Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big) \\ & \Big(2u(v-u)^{1-d/\alpha} \int_{\mathbb{R}^d} \widehat{\varphi}(y_1(T))\overline{\widehat{\psi}(y_1(T))} \frac{e^{-|y|^\alpha}}{|y|^\alpha} dy \\ & - \int_{v-u}^{v+u} \int_{\mathbb{R}^d} \widehat{\varphi}(y_2(T))\widehat{\psi}(y_2(T)) \frac{e^{-|y|^\alpha}}{|y|^\alpha} r^{1-d/\alpha} dr dy \Big). \end{aligned} \tag{2.18}$$

To obtain a limit in (2.18) we put $F_T = T^{2-d/2\alpha}$, and therefore

$$\begin{aligned} \lim_{T\to\infty} B_T(s,t) &= \frac{V\widehat{\varphi}(0)\overline{\widehat{\psi}(0)}}{4(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-|y|^\alpha}}{|y|^\alpha} dy \Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big) \\ & \Big(2u(v-u)^{1-d/\alpha} + \frac{(v-u)^{2-d/\alpha} - (v+u)^{2-d/\alpha}}{2-d/\alpha} \Big). \end{aligned} \tag{2.19}$$

But

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{-|y|^\alpha}}{|y|^\alpha} dy = \frac{\Gamma(d/\alpha - 1)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)}, \quad d > \alpha,$$

$$\begin{aligned} & \Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big) u(v-u)^{1-d/\alpha} \\ &= \frac{t^{4-d/\alpha} + s^{4-d/\alpha} - (4-d/\alpha)s(t-s)^{3-d/\alpha} - (t-s)^{4-d/\alpha}}{(2-d/\alpha)(3-d/\alpha)(4-d/\alpha)}, \quad d < 2\alpha, \end{aligned}$$

$$\Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big)(v-u)^{2-d/\alpha} = \frac{t^{4-d/\alpha} + s^{4-d/\alpha} - (t-s)^{4-d/\alpha}}{(3-d/\alpha)(4-d/\alpha)},$$

and

$$\Big(\int_0^s du \int_s^t dv + 2\int_0^s du \int_u^s dv \Big)(v+u)^{2-d/\alpha} = \frac{(t+s)^{4-d/\alpha} - t^{4-d/\alpha} - s^{4-d/\alpha}}{(3-d/\alpha)(4-d/\alpha)}.$$

Putting these results together and letting $h = 3 - d/\alpha$ we obtain (2.12) from (2.19). □

The following simple argument now proves Theorem 2.1.

Proof of Theorem 2.1:

Theorems 2.2 and 2.3 imply that the functions (2.1) and (2.2), which appear as the temporal parts of the limits in (2.9) and (2.12), are non-negative definite (as limits of non-negative definite functions). Therefore by a well-known result (see e.g. [42]) there exist real centered Gaussian processes η_h and σ_h having the functions (2.1) and (2.2) as covariances, respectively. □

Remark 2.4. (a) For the systems with initial particles distributed according to a uniform Poisson random field on $\mathbb{R}^d$ and no immigration, the normings for convergence of the occupation time fluctuations are $F_T = T^{(2-d/\alpha)/2}$ without branching, and $F_T = T^{(3-d/\alpha)/2}$ with branching [4]. Since the corresponding normings for the immigration systems are of higher orders ($F_T = T^{(3-d/\alpha)/2}$ and $F_T = T^{(4-d/\alpha)/2}$, respectively), the effect of the initial particles on the occupation time fluctuations of the immigration systems vanishes as $T \to \infty$. Hence Theorems 2.2 and 2.3 also hold with the initial particles included.
(b) In Theorem 2.2.(2) the temporal structure of the limit covariance corresponds to Brownian motion with change of time $t \mapsto t^2$, which does not have long-range dependence. The reason for giving this result is that the order of the norming was used in the proof of Theorem 2.3.

We consider next the branching and immigration process N itself (with no initial particles) and its normed fluctuation process $Y_T = \{Y_T(t), t \geq 0\}$, defined by

$$\langle Y_T(t), \varphi\rangle = \frac{1}{F_T}(\langle N_{Tt}, \varphi\rangle - E\langle N_{Tt}, \varphi\rangle), \quad t \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Theorem 2.5. *For* $(1 =)\, d < \alpha$, *with* $F_T = T^{(2-d/\alpha)/2}$, *and for any* $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,

$$\begin{aligned}&\lim_{T\to\infty} Cov(\langle Y_T(s), \varphi\rangle, \langle Y_T(t), \psi\rangle))\\ &= \langle\lambda, \varphi\rangle\langle\lambda, \psi\rangle \frac{V\Gamma(2-h)}{2\pi\alpha(h-1)h}[(s+t)^h - (t-s)^h - 2hs(t-s)^{h-1}], \quad s \leq t, \qquad (2.20)\end{aligned}$$

where $h = 2 - d/\alpha, h \in (1, 3/2]$.

For $\alpha = 2$ this result agrees with an immigration superprocess fluctuation limit in [39] (Theorem 1.10). The temporal part of the limit in (2.20) is the function (1.5), and therefore a real centered Gaussian process with this function as covariance exists. Theorem 2.5 is proved similarly as the previous theorems.

3. Fluctuations of the demographic variation process

An objective of this section is to show that different types of particle systems may lead to the same long-range dependence process. We consider the case of sub-fBm.

We recall the definition of the demographic variation process in a special case (see [23] for a general setup and details, and [24] for a multitype context). We consider the branching particle system described in Section 2, but now we suppress the immigration. The measure-valued process $N = \{N_t, t \geq 0\}$ is defined as before. We decompose N_t as $N_t = N_t^I + N_t^{II}$, where the processes $N^I = \{N_t^I, t \geq 0\}$ and $N^{II} = \{N_t^{II}, t \geq 0\}$ are constructed as follows. For each initial particle, if it splits into two particles, then only one of them (chosen at random) is retained, and if it dies, then it is replaced by a new particle at the death site and this particle lives forever (without branching) moving according to the α-stable process. If the initial particle splits into two, then the same is done with the retained particle, and so on. This defines the process N^I, which is just a Poisson system of independent (non-branching) particles. N^I is called the "basic population process" of the branching particle system. The process N^{II} is defined by

$N^{II}_t = N_t - N^I_t$ for each t. Since N^{II} measures the difference between the branching particle system and the basic population process due to branchings and deaths in the population, it is called "demographic variation process" of the branching system. Note that the atoms of N^{II}_t have charge 1 if they belong to N_t, and they have charge -1 if they belong to N^I_t (cancellations of charges due to particles at the same site at the same time have probability zero). The processes N and N^I are Markov. N^{II} is not Markov because a particle that was retained and has not branched up to time t does not appear in N^{II}_t but it produces atoms of N^{II} in the future when it splits or dies.

Recall that fBm and sub-fBm with $h > 1$ are obtained from rescaled occupation time fluctuations of the processes N^I and N, respectively [4]. We will show that the rescaled occupation time fluctuations of the process N^{II} also lead to sub-fBm with $h > 1$.

The rescaled occupation time process $L_T = \{L_T(t), t \geq 0\}$ of N^{II} is defined by

$$\langle L_T(t), \varphi\rangle = \int_0^{Tt} \langle N^{II}_s, \varphi\rangle ds = T\int_0^t \langle N^{II}_{Ts}, \varphi\rangle ds, \quad t \geq 0,\ \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{3.1}$$

and we consider the normed fluctuation process $X_T = \{X_T(t), t \geq 0\}$ defined by

$$\langle X_T(t), \varphi\rangle = \frac{1}{F_T}\langle L_T(t), \varphi\rangle, \quad t \geq 0, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \tag{3.2}$$

(in this special case, $EL_T(t) = 0$; see below).

The covariances for the present setup are obtained by specializing the model of [23] as follows (with the notation of [23]). Let $\gamma = 1$ (initial Poisson intensity), $\beta = 0$ (no immigration), $p_0 = p_2 = \frac{1}{2}$ (critical binary branching). Hence $\alpha = 0$ (α is the Malthusian parameter in [23]), $m_1 = 1, m_2 = 1$ (mean and second factorial moment of the branching law). We set the branching rate $V = 2$ (so that $Vp_0 = 1$, which simplifies some formulas, but the results are valid for any $V > 0$). In [23] the particle motion is Brownian motion but the results hold also for α-stable process ([24] deals with multitype α-stable motions). Then from [23] (Equations (14) and (16)) we obtain

$$Cov(\langle N_u, \varphi\rangle, \langle N_v, \psi\rangle) = \langle \lambda, \varphi\mathcal{T}_{v-u}\psi\rangle + 2\int_0^u \langle \lambda, \varphi\mathcal{T}_{v-u+2r}\psi\rangle dr, \ \ u \leq v, \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{3.3}$$

and

$$\begin{aligned} Cov(\langle N^{II}_u, \varphi\rangle, \langle N^{II}_v, \psi\rangle) &= 2(1-e^{-u})\langle \lambda, \varphi\mathcal{T}_{v-u}\psi\rangle + 2\int_0^u \langle \lambda, \varphi\mathcal{T}_{v-u+2r}\psi\rangle dr \\ &\quad -2e^{-u}\int_0^u e^r\langle \lambda, \varphi\mathcal{T}_{v-u+2r}\psi\rangle dr. \end{aligned} \tag{3.4}$$

($EL_T(t) = 0$ because $EN^{II}_t = 0$, by [23], Equation (15)).

Theorem 3.1. *For $\alpha < d < 2\alpha$, with $F_T = T^{(3-d/\alpha)/2}$, and for any $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\begin{aligned} &\lim_{T\to\infty} Cov(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\ &= \langle \lambda, \varphi\rangle\langle \lambda, \psi\rangle \frac{\Gamma(2-h)}{2^{d-2}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}\left(s^{h+1} + t^{h+1} - \frac{1}{2}[(s+t)^h + (t-s)^h]\right), \quad s \leq t, \end{aligned} \tag{3.5}$$

where $h = 3 - d/\alpha, \quad h \in (1, 2)$.

Proof: We have from (3.3) and (3.4),

$$Cov(\langle N^{II}_u, \varphi\rangle, \langle N^{II}_v, \psi\rangle) = Cov(\langle N_u, \varphi\rangle, \langle N_v, \psi\rangle) + (1-2e^{-u})H_1(u, v) - 2H_2(u, v), \tag{3.6}$$

where

$$H_1(u,v) = \langle \lambda, \varphi \mathcal{T}_{v-u}\psi\rangle, \quad u \le v, \tag{3.7}$$

$$H_2(u,v) = e^{-u}\int_0^u e^r \langle \lambda, \varphi \mathcal{T}_{v-u+2r}\psi\rangle dr, \quad u \le v. \tag{3.8}$$

The covariance of the process X_T is given by

$$Cov(\langle X_T(s),\varphi\rangle, \langle X_T(t),\psi\rangle) = \frac{T^2}{F_T^2}\int_0^s du \int_0^t dv\, Cov(\langle N_{Tu}^{II},\varphi\rangle, \langle N_{Tv}^{II},\psi\rangle), \tag{3.9}$$

and a corresponding formula holds for the covariance of the occupation time fluctuation of the rescaled process N. In the case of N the covariance converges to the right-hand side of (3.5) for $\alpha < d < 2\alpha$, with $F_T = T^{(3-d/\alpha)/2}$, as $T \to \infty$ [4]. Therefore we see from (3.6)-(3.9) that in order to prove the theorem it suffices to show that

$$\lim_{T\to\infty} T^{d/\alpha-1}\int_0^s du \int_0^t dv[(1-2e^{-Tu})H_1(Tu,Tv) - 2H_2(Tu,Tv)] = 0, \tag{3.10}$$

and for this it is enough to prove that

$$\lim_{T\to\infty} T^{d/\alpha-1} H_1(Tu,Tv) = 0, \tag{3.11}$$

$$\lim_{T\to\infty} T^{d/\alpha-1} H_2(Tu,Tv) = 0. \tag{3.12}$$

Now, (3.11) is immediate from (3.7) because, by the scaling property the α-stable process on $\mathbb{R}^d$,

$$\mathcal{T}_t\varphi = O(t^{-d/\alpha}) \text{ as } t \to \infty. \tag{3.13}$$

From (3.8) and again by (3.13) we have $H_2(Tu,Tv) = O(T^{-d/\alpha})$ as $T \to \infty$, so (3.12) follows. □

Remark 3.2. (a) Note that (3.10) holds for any values of d and α.
(b) We have seen that the occupation time fluctuations of the processes N and N^{II} both lead to sub-fBm, although these processes are quite different. N is Markov and N^{II} is not. On the other hand, the extinction of individual families caused by the critical branching produces more and more particles which contribute to the process N^I, and therefore to the negative atoms of N^{II}. We now explain why the occupation time fluctuations of N and N^{II} have the same asymptotic behavior. Any region where particles are going extinct (and becoming particles with charge -1) is repopulated by particles (with charge 1) coming from other regions of space by transience of the motion (since $d > \alpha$) which have not died yet (in fact these two effects balance each other and produce an equilibrium state of the process N in the large time limit [27]), and since $(3 - d/\alpha)/2 > 1/2$ (because $d < 2\alpha$), the effect of the process N^I on the occupation time fluctuation limit of N^{II} becomes negligible because the norming for the occupation time fluctuation limit of N^I is $T^{1/2}$ for $d > \alpha$.
(c) Another particle system (without branching) which also leads to sub-fBm is given in [4].

4. Properties of the long-range dependence processes η_h and σ_h

In Theorem 2.1 we established the existence of the processes η_h and σ_h. In this section we give some of their properties.

Theorem 4.1. *The process η_h has the following properties:*

(a) η_h *is* $\frac{h+1}{2}$*-self-similar.*

(b)

$$C_{\eta_h}(s,t) > 0, \quad s,t > 0.$$

(c)

$$E(\eta_h(t) - \eta_h(s))^2 = 2(hs+t)(t-s)^h, \quad s \le t,$$

hence

$$2(h+1)s(t-s)^h \le E(\eta_h(t) - \eta_h(s))^2 \le 2(h+1)t(t-s)^h, \quad s \le t.$$

(d) *Hölder continuity:* η_h *has a continuous version, and for each* $\varepsilon \in (0, h/2]$ *and each* $L > 0$ *there exists a random variable* $K_{\varepsilon,L}$ *such that*

$$|\eta_h(t) - \eta_h(s)| \le K_{\varepsilon,L}|t-s|^{\frac{h}{2}-\varepsilon}, \quad s,t \in [0,L], \quad a.s.$$

(e) *Let*

$$R_{\eta_h}(u,v,s,t) = E(\eta_h(v) - \eta_h(u))(\eta_h(t) - \eta_h(s)), \quad 0 \le u < v \le s < t,$$

Then

$$R_{\eta_h}(u,v,s,t) > 0$$

and

$$R_{\eta_h}(u,v,s+\tau,t+\tau)\tau^{2-h} \to \frac{1}{2}(h-1)h(h+1)(t-s)(v^2-u^2) \quad as \quad \tau \to \infty.$$

(f) η_h *is not a Markov process.*

(g) η_h *is not a semimartingale.*

Theorem 4.2. *The process* σ_h *has the following properties*:

(a) σ_h *is* $\frac{h+1}{2}$*-self-similar.*

(b)

$$C_{\sigma_h}(s,t) > 0, \quad s,t > 0. \tag{4.1}$$

(c)

$$E(\sigma_h(t)-\sigma_h(s))^2 = -2^{h-1}(s^{h+1}+t^{h+1}) + \frac{1}{2}(s+t)^{h+1} + \frac{1}{2}[(2h-1)s+3t](t-s)^h, \ s \le t, \tag{4.2}$$

and there exist a positive constant c_h *such that*

$$c_h s(t-s)^h \le E(\sigma_h(t) - \sigma_h(s))^2 \le (h+1)t(t-s)^h. \tag{4.3}$$

(d) *Hölder continuity:* σ_h *has a continuous version, and for each* $\varepsilon \in (0, h/2]$ *and each* $L > 0$ *there exists a random variable* $K_{\varepsilon,L}$ *such that*

$$|\sigma_h(t) - \sigma_h(s)| \le K_{\varepsilon,L}|t-s|^{\frac{h}{2}-\varepsilon}, \quad s,t \in [0,L], \quad a.s.$$

(e) *Let*

$$R_{\sigma_h}(u,v,s,t) = E(\sigma_h(v) - \sigma_h(u))(\sigma_h(t) - \sigma_h(s)), \quad 0 \le u < v \le s < t,$$

Then

$$R_{\sigma_h}(u,v,s,t) > 0, \tag{4.4}$$

and

$$R_{\sigma_h}(u,v,s+\tau,t+\tau)\tau^{3-h} \to \frac{1}{6}(h-1)h(h+1)(2-h)(t-s)(v^3-u^3) \quad as \quad \tau \to \infty. \tag{4.5}$$

(f) σ_h *is not a Markov process.*

(g) σ_h *is not a semimartingale.*

Remark 4.3. The decay rates τ^{h-2} and τ^{h-3} of the covariances of increments for η_h and σ_h on intervals separated by distance τ, which characterize their long-range dependence, are the same as for fBm and sub-fBm, respectively [4].

Since the proofs for η_h and σ_h are analogous, we will give only those for σ_h, which involve additional work.

Proof of Theorem 4.2:

Denote for brevity $C(s,t) = C_{\sigma_h}(s,t)$ and $R(u,v,s,t) = R_{\sigma_h}(u,v,s,t)$.

(a) The self-similarity, i.e., $\{\sigma_h(at), t \ge 0\} \stackrel{d}{=} \{a^{(h+1)/2}\sigma_h(t), t \ge 0\}$ for any $a > 0$, is obvious from the form of $C(s,t)$ given in (2.2).

(b) From (2.2) we have $C(0,t) = 0$, $C(t,t) = (2 - 2^{h-1})t^{h+1}$ for $t > 0$, and

$$\frac{\partial}{\partial s}C(s,t) = (h+1)s^h \left(1 - \frac{1}{4}m\left(\frac{t}{s}\right)\right), \quad s \le t,$$

where

$$m(x) = (x+1)^h - (x-1)^h - 2h(x-1)^{h-1}, \quad x \ge 1.$$

We will show that $0 < m(x) < 4$ for $x > 1$, which implies (4.1). Since $m(1) = 2^h < 4$ and $m(x) \to 0$ as $x \to \infty$, it suffices to prove that $m(x)$ is decreasing. We have

$$m'(x) = h[(x+1)^{h-1} - (x-1)^{h-1} - 2(h-1)(x-1)^{h-2}].$$

By the mean value theorem, $(x+1)^{h-1} - (x-1)^{h-1} = 2(h-1)x_0^{h-2}$ for some $x_0 \in (x-1, x+1)$. Hence, since $h < 2$,

$$m'(x) = 2h(h-1)(x_0^{h-2} - (x-1)^{h-2}) < 0.$$

(c) (4.2) is obtained from (2.2). By convexity, $(s+t)^{h+1} \le 2^h(s^{h+1} + t^{h+1})$, hence

$$-2^{h-1}(s^{h+1} + t^{h+1}) \le -\frac{1}{2}(s+t)^{h+1},$$

and then

$$E(\sigma_h(t) - \sigma_h(s))^2 \le \frac{1}{2}[(2h-1)s + 3t](t-s)^h \le (h+1)t(t-s)^h,$$

so we have the upper bound in (4.3).

For the lower bound, let

$$f(x) = -2^h(1 + x^{h+1}) + (1+x)^{h+1} + (1-x)^h[(2h-1)x + 3], \quad 0 \le x \le 1,$$

so

$$E(\sigma_h(t)-\sigma_h(s))^2=\frac{1}{2}t^{h+1}f\left(\frac{s}{t}\right),\quad s\le t. \tag{4.6}$$

We have

$$\frac{f(x)}{(1-x)^h}=(2h-1)x+3+g(x),$$

where

$$g(x)=\frac{-2^h(1+x^{h+1})+(1+x)^{h+1}}{(1-x)^h}.$$

Since $g(x)\to 0$ as $x\to 1$, then

$$\frac{f(x)}{x(1-x)^h}\to 2(h+1)\quad\text{as}\quad x\to 1,$$

and since $f(x)>0$ for $x\in[0,1)$, there is a constant $c_h>0$ such that

$$f(x)>c_h x(1-x)^h\quad\text{for all}\quad x\in[0,1].$$

The lower bound in (4.3) then follows from (4.6).

(d) The Hölder continuity of σ_h follows from the right-hand inequality in (4.3) by Kolmogorov's criterion.

(e) From (2.2) we obtain, fixing u, s and t,

$$R(u,v,s,t)=\frac{1}{4}F_{u,s,t}(v),\quad u\le v\le s, \tag{4.7}$$

where

$$\begin{aligned}F_{u,s,t}(v) \;=\;& (t-u)^h[(2h-1)u+3t]-(s-u)^h[(2h-1)u+3s]\\ &-(t-v)^h[(2h-1)v+3t]+(s-v)^h[(2h-1)v+3s]\\ &+(t+u)^{h+1}-(s+u)^{h+1}-(t+v)^{h+1}+(s+v)^{h+1}.\end{aligned}$$

Then $F_{u,s,t}(u)=0$ and

$$F'_{u,s,t}(v)=(h+1)G_{s,t}(v),$$

where

$$\begin{aligned}G_{s,t}(v) \;=\;& (t-v)^h+(t-v)^{h-1}2hv-(s-v)^h-(s-v)^{h-1}2hv-(t+v)^h+(s+v)^h,\\ & \qquad\qquad v\le s\le t.\end{aligned}$$

We have $G_{s,s}(v)=0$ and

$$\frac{\partial}{\partial t}G_{s,t}(v)=h(t-v)^{h-2}H_t(v),$$

where

$$H_t(v)=t-v-\frac{(t+v)^{h-1}}{(t-v)^{h-2}}+2(h-1)v,\quad 0\le v\le t.$$

(Note that $H_t(v)$ does not depend on s). Then $H_t(0)=0$ and

$$H'_t(v)=2h-3-(h-1)\left(\frac{t+v}{t-v}\right)^{h-2}+(2-h)\left(\frac{t+v}{t-v}\right)^{h-1}.$$

But $\left(\frac{t+v}{t-v}\right)^{h-2} < 1$ and $\left(\frac{t+v}{t-v}\right)^{h-1} > 1$ since $1 < h < 2$, therefore

$$H_t'(v) > 2h - 3 - (h-1) + (2-h) = 0.$$

Hence $F_{u,s,t}(v) > 0, u < v$, so (4.4) follows from (4.7).

For the proof of (4.5) we write $R(\tau) = R(u, v, s+\tau, t+\tau)$ and $a = t-u, b = t-v, c = s-v, d = s-u, e = t+u, f = t+v, g = s+v, k = s+u$. We have from (4.7)

$$\begin{aligned} R(\tau) \;\; = \;\; & \tau^{h+1}\Bigg\{\frac{3}{4}\Bigg[\left(1+\frac{a}{\tau}\right)^{h+1} - \left(1+\frac{b}{\tau}\right)^{h+1} + \left(1+\frac{c}{\tau}\right)^{h+1} - \left(1+\frac{d}{\tau}\right)^{h+1}\Bigg] \\ & + \frac{1}{4}\Bigg[\left(1+\frac{e}{\tau}\right)^{h+1} - \left(1+\frac{f}{\tau}\right)^{h+1} + \left(1+\frac{g}{\tau}\right)^{h+1} - \left(1+\frac{k}{\tau}\right)^{h+1}\Bigg] \\ & + \frac{h+1}{2\tau}\Bigg[\left(1+\frac{a}{\tau}\right)^{h} u - \left(1+\frac{b}{\tau}\right)^{h} v + \left(1+\frac{c}{\tau}\right)^{h} v - \left(1+\frac{d}{\tau}\right)^{h} u\Bigg]\Bigg\}. \end{aligned}$$

Then (4.5) is obtained by four successive applications of L'Hôpital's rule together with the following equalities (we omit details):

(i) the first time,

$-a + b - c + d = 0, \; -e + f - g + k = 0,$

(ii) the second time,

$3(a^2 - b^2 + c^2 - d^2) = 6(t-s)(v-u),$

$e^2 - f^2 + g^2 - k^2 = -2(t-s)(v-u),$

$-4(-au + bv - cv + du) = -4(t-s)(v-u),$

(iii) the third time,

$3(-a^3 + b^3 - c^3 + d^3) = -9(t-s)(v-u)(s+t-v-u),$

$-e^3 + f^3 - g^3 + d^3 = 3(t-s)(v-u)(s+t+v+u),$

$-b(a^2u - b^2v + c^2v - d^2u) = 6(t-s)(v-u)(t+s-2(u+v)),$

(iv) the fourth time,

$3(a^4 - b^4 + c^4 - d^4) = 6(t-s)(v-u)[2(s^2+st+t^2) - 3(t+s)(v+s) + 2(u^2+vu+v^2)],$

$e^4 - f^4 + g^4 - k^4 = -2(t-s)(v-u)[2(s^2+st+t^2) + 3(t+s)(v+s) + 2(u^2+uv+v^2)],$

$-8(a^3u + b^3v - c^3v + d^3u) = -8(t-s)(v-u)[s^2+st+t^2 - 3(t+s)(v+s) + 3(u^2+uv+v^2)],$

and the sum of the last three terms is equal to $-16(t-s)(v^3-u^3)$.

(f) The covariance given by (2.2) does not have the triangular property which is necessary for a Gaussian process to be Markovian (see e.g. [42]).

(g) The following result is contained in [3] (Lemma 2.1):

Let $\chi = \{\chi(t), t \in [0,1]\}$ *be a real, continuous, centered Gaussian process such that for some* $h \in (0,2), h \neq 1$, *and some positive constants* c_1 *and* c_2,

$$c_1(t-s)^h \leq E(\chi(t) - \chi(s))^2 \leq c_2(t-s)^h, \quad s \leq t.$$

Then χ *is not a semimartingale*

Using this result with (4.3) shows that σ_h is not a semimartingale on $[\delta, 1]$ for any $\delta \in (0,1)$. □

5. Some comments

Functional convergence with generalized Gaussian limits for the occupation time fluctuations of the particle systems without immigration in the cases of long-range dependence (related to covariances (1.1) and (1.2)) are proved in [5]. The proofs involve tightness, and a space-time random field method for identification of a unique limit which is specially useful for non-Markov processes. The (weak) convergence takes place in the space of continuous functions $C([0,\tau],\mathcal{S}'(\mathbb{R}^d))$ for any $\tau > 0$, where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. It is well known that this space of distributions is appropriate for convergence results for this type of systems due to the nuclear property of $\mathcal{S}(\mathbb{R}^d)$. The same approach may be used for proving functional limit theorems corresponding to Theorems 2.2, 2.3 and 2.5, obtaining $\mathcal{S}'(\mathbb{R}^d)$-valued centered Gaussian processes with covariances given by the right hand sides of (2.9), (2.12) and (2.20). The functional version of Theorem 2.5 requires the Skorohod space $D([0,\tau],\mathcal{S}'(\mathbb{R}^d))$. The $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian processes $X = \{X(t), t \geq 0\}$ with covariances given by the right hand sides of (2.9) and (2.12) can be represented as follows: For Theorem 2.2.(1),

$$X(t) = \left(\frac{\Gamma(2-h)}{\pi\alpha(h-1)h(h+1)}\right)^{1/2} \lambda\eta_h(t), \quad t \geq 0, \quad h \in (1, 3/2].$$

where η_h is the real Gaussian process with covariance (2.1). For Theorem 2.3,

$$X(t) = \left(\frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)(h-1)h(h+1)}\right)^{1/2} \lambda\sigma_h(t), \quad t \geq 0, \quad h \in (1, 2).$$

where σ_h is the real Gaussian process with covariance (2.2). In both cases the spatial structure is simple (Lebesgue measure) and the temporal structure is complicated (with long-range dependence).

The real Gaussian process with covariance (1.5) and the $\mathcal{S}'(\mathbb{R}^d)$-valued Gaussian process with covariance given by the right hand side of (2.20) are not Markov although the approximating process Y_T is Markov. This is an example of the fact that the Markov property is not necessarily preserved under weak convergence.

We noted in Remark 4.3 that the decay rates of the covariances of increments for η_h and σ_h are the same as for fBm and sub-fBm. Therefore the immigration does not change the decay rate of the long-range dependence of the occupation time fluctuations of the particle systems; only the branching accounts for the difference. On the other hand, the immigration increases the sizes of the fluctuations by $T^{1/2}$ in each case (see Remark 2.4.(a)).

This paper and [4, 5] deal with occupation time fluctuations in the cases of long-range dependence: $d < \alpha$ for the non-branching system, and $\alpha < d < 2\alpha$ for the branching system. There are also functional limit theorems in the cases where the temporal structure of the limit processes has independent increments (without immigration): $d \geq \alpha$ for the non-branching system, and $d \geq 2\alpha$ for the branching system [6]. The main features of the results for the branching system are: for $d = 2\alpha$ with $F_T = (T \log T)^{1/2}$, the limit has the form const.$\lambda\beta$ where $\beta = \{\beta(t), t \geq 0\}$ is Bm, and for $d > 2\alpha$ with $F_T = T^{1/2}$, the limit is a truly generalized process, an $\mathcal{S}'(\mathbb{R}^d)$-valued Wiener process (see [2] for $\mathcal{S}'(\mathbb{R}^d)$-valued Wiener processes). Thus there is a jump in the size of the fluctuations at the critical case $d = 2\alpha$, but the spatial structure is still simple (Lebesgue measure). The size of the fluctuations is the "classical" (central limit theorem) one for $d > 2\alpha$, and the spatial structure becomes complicated ($\mathcal{S}'(\mathbb{R}^d)$-valued). The critical case separates two very different types of behaviors, both spatially and temporally. The same thing occurs similarly for the non-branching system, the critical case being $d = \alpha$ (recall also the result of Theorem 2.2.(2) and Remark 2.4(b)). It should be possible to prove analogous functional limit theorems for the systems with immigration using the methods of [6].

The causes of long-range dependence of the occupation time fluctuations in the systems with or without branching and no immigration are the types of the particle motion: strict recurrence ($d < \alpha$) in the system without branching, and strict weak transience ($\alpha < d < 2\alpha$) in the branching system. In the first case the particles return infinitely often and at arbitrarily large times to any given bounded interval, each time adding a random amount of time to the occupation of the interval; and in the second case any given ball is visited infinitely often and at arbitrarily large times by clans, each time adding a random amount to the occupation of the ball (a "clan" is a family of particles with eventually backwards coalescing paths); see [4] and [5] for more details, and [48] for "clan recurrence" of branching particle systems in equilibrium in the case $\alpha = 2$. As noted above, long-range dependence disappears at the borders: $d = \alpha$ (critical recurrence) for the non-branching system, and $d = 2\alpha$ (critical weak transience) for the branching system. We have seen that the immigration increases the size of the occupation time fluctuations, and it also modifies the form of long-range dependence (e.g., from covariance (1.2) to covariance (1.4) in the branching system), but not the decay rate. We have also seen that the immigration causes long-range dependence on the branching particle process N itself. Another possible source of long-range dependence in branching systems could be a particle lifetime distribution with long tail (see [55]).

Non-Gaussian processes and random fields with long-range dependence also appear in applications (see e.g. [30, 37, 38, 49, 50]. Long-range dependence non-Gaussian processes are also found in the context of branching particle systems . In the branching system without immigration, for $\alpha = d$ the occupation time process with $F_T = T$ obeys a functional ergodic theorem, and the corresponding occupation time fluctuation limit is a centered measure-valued process [6, 52] (see also [21, 35] in the context of superprocesses). This limit is a finite-variance non-Gaussian process with long-range dependence. The covariance of increments on intervals separated by distance τ decays like τ^{-1}. Thus, there is a continuity of the results for $\alpha < d < 2\alpha$ as $\alpha \nearrow d$ in the forms of the norming and the long-range dependence, but at the border $\alpha = d$ the limit fluctuation process becomes non-Gaussian. A similar effect should also occur for the branching system with immigration. In the branching systems with $\alpha < d < 2\alpha$ (with or without immigration) the fluctuation limit processes are Gaussian because the branching law has finite variance. With a (critical) branching law in the domain of attraction of a stable law with exponent $1 + \beta, (0 < \beta < 1)$, one finds that for $\alpha/\beta < d < \alpha(1 + \beta)/\beta$ and $F_T = T^{(2+\beta-d\beta/\alpha)/(1+\beta)}$ (without immigration), the occupation time fluctuation limit process is a centered measure-valued $(1 + \beta)$-stable process with long-range dependence [7].

We have considered branching systems with the assumption of critical binary branching, which simplifies proofs. It would be interesting to study long-range dependence of fluctuations for more general branching systems. In the case of general critical finite-variance branching, the results should be the same with V multiplied by the second factorial moment of the branching law. A different case from those we have looked at is subcritical branching systems with immigration (which have equilibrium states). Also, long range-dependence in occupation time fluctuations of multitype branching particle systems would be interesting to study (see [40] for fluctuations of such systems).

Occupation time fluctuations of the particle systems without immigration yield fBm and sub-fBm only with $h > 1$ (aggregation) [4, 5]. In order to obtain these processes with $h < 1$ (intermittency) from occupation times of particle systems, it would be necessary to introduce some kind of interaction between particles (e.g., repulsion). This has not been tried.

Occupation times are relevant for some applications. For example, in air pollution models (see e.g. [25, 26]) the occupation time of a particle system may represent the accumulated exposure to suspended particulate matter.

Some models in financial mathematics (e.g. Black-Scholes models) use stochastic calculus based on Bm or on fBm as guiding processes, and there is discussion among specialists on which process should be used. Since sub-fBm is intermediate between Bm and fBm (in the sense that increments are more weakly correlated and their covariance decays faster than for fBm), it may be useful to investigate possible applications of sub-fBm in this field. This would require a stochastic calculus with respect to sub-fBm, in particular a representation of sub-fBm as stochastic integral with respect to Bm (such a representation can be obtained from [17]). Also, long-memory stochastic volatilities could perhaps be represented by sub-fBm or other long-range dependence processes such as those discussed above. Branching mechanisms have been introduced for stock price models [19], and immigration may also have potential applications in finance.

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