OPTIMAL CLOSED-FORM SOLUTION TO AN UNCONSTRAINED MULTIPERIOD MEAN-VARIANCE PORTFOLIO PROBLEM

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Abstract

The present paper offers a closed-form solution to an unconstrained multiperiod mean-variance problem when the investor's portfolio consists of a single stock and bond and where only fairly general conditions are imposed on these assets. As the reader will see, among the advantages of the proposed solution one finds that it is general enough to allow for the incorporation of time dependence in modelling the rate of return, as well as dependence, if one so wishes, on exogenous variables, such as economic factors that might have the property to improve substantially our ability to assess future rate of return. Our proof is basically constructive and our argumentation yields as a corollary a security market line result in a multiperiod capital asset pricing model.

Key Words: multi-period setting, mean-variance analysis, optimal portfolio strategy

1 INTRODUCTION

Half a century ago portfolio management was at a turning point with the publication of H. M. Markowitz's paper [12]. He pioneered the first rigorous treatment of an investor's dilemma, that is, how to attain higher profits while downsizing risk. For his mean-variance approach in portfolio selection, H. Markowitz received the 1990 Nobel Prize in Economics (shared with M.H. Miller and W. Sharpe).

Ever since then, mean-variance analysis has been a topic which generates much interest among researchers as well as practitioners. In the first stage, single-period analytical solutions were developed by Markowitz [12] and Merton [14] (1997 Nobel prize in Economics shared with M. S. Scholes) and this naturally led to the original capital asset pricing model (CAPM). Related research in the dynamic multiperiod case was done mainly by Tobin [24], Mossin [16], Samuelson [20], Fama [6], Hakansson [8] [9], Stevens [23], Markowitz [13], Elton and Gruber [5], Schweizer [21] and Pliska [18]. A detailed historical review of these and other contributions to the subject can be found in Steinbach [22], which also contains a very extensive bibliography of more than 200 references. For the continuous time version of the problem, the reader should consult Pliska [17], Richardson [19], Merton [15], Duffie and Richardson [4], Karatzas and Shreve [10] and Korn and Korn [11].

The present paper offers a closed-form solution to the unconstrained multiperiod mean-variance problem when the investor's portfolio consists of a single stock and bond and where only fairly general conditions are imposed on these assets. As the reader will see, among the advantages of the proposed solution one finds that it is general enough to allow for the incorporation of time dependence in modelling the rate of return, as well as dependence, if one so wishes, on exogenous variables, such as economic factors that might have the property to improve substantially our ability to assess future rate of return.

The paper is organized as follows. In section 2, an expression for the rate of return of the investor's portfolio at a given time is developed in terms of the excess rate of return of the stock over the constant interest rate and is then followed by the corresponding formulation of the multiperiod mean-variance optimization problem. We establish key conditional mean, conditional variance and conditional covariance properties of a specific strategy. This strategy is then shown to be the optimal mean-variance solution. In the process, we supply sufficient conditions for the boundedness of the mean-variance tradeoff, which is a key hypothesis in the treatment of this problem given in Schweizer [21]. Said sufficient conditions also allow us to avoid the uniform non degeneracy condition (noted ND) used there, by replacing it with the weakest requirement possible in the context of the mean-variance problem, namely that $\{R_n\}$ forms a submartingale for the entire duration of the investment. This requirement corresponds to some optimism on the part of the investor in the sense that the market is expected to yield a favorable rate of return for the risky asset at every portfolio reshuffling time. Note that condition (ND) allows Schweizer [21] to draw attention to a fundamental link between the existence of a solution to the mean-variance problem and the topological closure of the set of all processes of discounted gains from trade. Our proof is basically constructive and offers no such link; on the other hand, our argumentation yields as a corollary, a new, generalized multiperiod capital asset pricing model. A wide class of examples are presented in section 3, the examples being chosen according to the following criteria : simplicity of form of the solution, reduced complexity of the statistical estimation of parameters, computational accuracy and efficiency in real-time calculations. Section 4 is devoted to the proofs of the main results presented in section 2. The paper concludes in section 5 with general remarks and indications for the line of investigation into future studies.

2 MULTIPERIOD SETTING AND SOLUTION TO THE CORRESPONDING MEAN-VARIANCE PROBLEM

We place ourselves in a context where a small investor holds a portfolio consisting of one risky asset and one riskless asset in a frictionless market. We use the term small investor in the sense that the composition of the portfolio held by this investor at any given time does not affect future prices in the market.

Let n be the terminal date, then for $k = 0, \ldots, n$, let P_k be the unitary value of the risky asset at time k, B_k be the unitary value of the riskless asset at time k and X_k the total value of the portfolio at time k. Let v_{k-1} be the total value of the shares of the risky asset held in hand just before time k, $R_k = \frac{P_k - P_{k-1}}{P_{k-1}}$ the rate of return of the risky asset at time k, $r = \frac{B_k - B_{k-1}}{B_{k-1}}$ the constant rate of return of the riskless asset at time k, then the wealth variation between time $k-1$ and k may be expressed in term of the excess rate of return $R_k - r$ at time k, as follows :

$$\begin{aligned} X_k - X_{k-1} &= v_{k-1}\left(\frac{P_k - P_{k-1}}{P_{k-1}}\right) + (X_{k-1} - v_{k-1})\left(\frac{B_k - B_{k-1}}{B_{k-1}}\right) \\ &= v_{k-1}R_k + (X_{k-1} - v_{k-1})r \\ &= v_{k-1}(R_k - r) + rX_{k-1}. \end{aligned}$$

Let $\omega_k = \frac{v_{k-1}}{X_{k-1}}$ be the fraction or weight of the portfolio allocated to the risky asset just before time k then we have can express the rate of return $\tilde{R_k}$ of the portfolio at time k by

$$\tilde{R_k} = \frac{X_k - X_{k-1}}{X_{k-1}} = \frac{v_{k-1}}{X_{k-1}}(R_k - r) + r = \omega_k(R_k - r) + r.$$

If $\omega_k \geq 1$ then we are in the presence of a leverage strategy and if $\omega_k \leq 0$ then this denotes a short-selling strategy. On the other hand if $0 \leq \omega_k \leq 1$ for all k then we will say that the portfolio is self-financed. Notice also that in the multiperiod setting the cumulative rate of return of the portfolio between a given time k and terminal date n can be expressed as $\Pi_{i=k}^{n}(1 + \widetilde{R}_i) - 1$.

All random variables in this paper are assumed to be built in a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. All other σ-field defined later will be contained in $\mathcal{F}$. Consider $\mathcal{F}_k$ the σ-field generated by $\{P_i, 0 \leq i \leq k\}$, $\mathcal{G}_k$ the σ-field generated by exogenous variables called signals $\{S_i, 0 \leq i \leq k\}$ and $\mathcal{H}_k = \mathcal{F}_k \vee \mathcal{G}_k$. Thus $\mathcal{H}_k$ consists of all information disposable to the agent up to time k concerning the history of the prices of the risky asset as well as economic indicators of the flow of the market . Therefore ω_k is considered to be $\mathcal{H}_{k-1}$-measurable, meaning that the fraction of wealth allocated to the risky asset is determined just before time k and based on information given up to time $k-1$.

The main objective is to develop a strategy that, given a desired expected cumulative rate of return $\Pi_{i=1}^{n}(1 + \widetilde{R}_i) - 1$ of the portfolio at terminal date n, minimizes the variance of this global rate of return.

For the following lemmas and theorems define $\sum_{i \epsilon \phi} \alpha_i = 0$ and $\Pi_{i \epsilon \phi}\alpha_i = 1$ for sums and products over the empty set. Furthermore, define recursively for $i = n-1, n-2, \ldots, 2, 1, 0$,

(2.1)
$$\tau_{n-1} = \frac{\mathbb{E}^2\left[R_n - r|\mathcal{H}_{n-1}\right]}{\mathbb{E}\left[(R_n - r)^2|\mathcal{H}_{n-1}\right]},$$

(2.2)
$$\tau_i = \frac{\mathbb{E}^2\left[\left(1 - \sum_{j=i+1}^{n-1}\tau_j\right)(R_{i+1} - r)|\mathcal{H}_i\right]}{\mathbb{E}\left[\left(1 - \sum_{j=i+1}^{n-1}\tau_j\right)(R_{i+1} - r)^2|\mathcal{H}_i\right]}.$$

We will show in the following that a portfolio with the weight associated to the risky asset defined for $k = 1, \ldots, n$ by

(2.3)
$$\omega_k = -\left((1+r) + \frac{\lambda_n}{2(1+r)^{n-k}\Pi_{i=1}^{k-1}(1+\widetilde{R}_i)}\right)\frac{\mathbb{E}\left[\left(1 - \sum_{i=k}^{n-1}\tau_i\right)(R_k - r)|\mathcal{H}_{k-1}\right]}{\mathbb{E}\left[\left(1 - \sum_{i=k}^{n-1}\tau_i\right)(R_k - r)^2|\mathcal{H}_{k-1}\right]}$$

when $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i) > 0$ has all the given properties to be an optimal solution to the unconstrained multiperiod mean-variance portfolio problem if we make a judicious choice of the constant λ_n. Motivation for the choice of (2.3) will be given in the proof of proposition 2.8.

Remark 2.1 *In* (2.1) *and* (2.2) *the form 0/0 is to be read as 0*

For the remainder of the article, we will make the general assumption that $\mathbb{E}\left[R_j^2\right] < \infty$ for all j, which will guarantee the almost sure finiteness of all the conditional expectations appearing in definitions (2.2), (2.3) and (2.10).

First, we shall exhibit sufficient conditions to safeguard against any one of the parameters τ_k and ω_k (as well as the optimal λ_k made explicit in (2.10) below) being ill-defined.

Lemma 2.2 *With the conventions above, there holds almost surely, for every* $i \in \{0, 1, \ldots, n-2\}$

(2.4)
$$0 \le \tau_i \le \mathbb{E}\left[1 - \sum_{j=i+1}^{n-1}\tau_j|\mathcal{H}_i\right] \le 1.$$

Lemma 2.3 *Under the hypothesis* $\mathbb{E}[\tau_0] > 0$*, the parameters appearing in formulas* (2.2)*,* (2.3) *and* (2.10) *are well-defined.*

The three following key lemmas give explicit expressions for the conditional mean, conditional variance and conditional covariance of the total rate of return of the portfolio from a given time up to the terminal date.

Lemma 2.4 *Let λ_n be a real number. Then for every $k = 1, ..., n$ the weights given by* (2.3) *associated with the optimal portfolio, satisfy*

$$(2.5)\quad \mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i)|\mathcal{H}_{k-1}\right] = (1+r)^{n-k+1}\,\mathbb{E}\left[1-\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right] - \frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right].$$

on all trajectories such that $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i) > 0$.

Lemma 2.5 *Let λ_n be a real number. Then for every $k = 1, ..., n$ the weights given by* (2.3) *associated with the optimal portfolio, satisfy*

$$(2.6)\quad \mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_{k-1}\right] = (1+r)^{2(n-k+1)}\,\mathbb{E}\left[1-\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right] + \frac{\lambda_n^2}{4\Pi_{i=1}^{k-1}(1+\widetilde{R}_i)^2}\mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right].$$

on all trajectories such that $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i) > 0$.

Lemma 2.6 *Let λ_n be a real number. Then for every $k = 1, ..., n$ and every arbitrary portfolio Q the weights given by* (2.3) *associated with the optimal portfolio P, satisfy*

$$\begin{aligned}(2.7)\quad & \mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right] \\ = {} & (1+r)^{2(n-k)+1}\left(\frac{\lambda_n}{2(1+r)^{n-k}\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left[1-\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right] \\ & + (1+r)^{2(n-k)+2}\,\mathbb{E}\left[1-\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right] \\ & - \frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right].\end{aligned}$$

on all trajectories such that $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P) > 0$.

Remark 2.7 *From lemmas 2.4, 2.5 and 2.6, we can establish*

$$\begin{aligned}(2.8)\quad & \mathbb{COV}\left(\Pi_{i=k}^{n}(1+\widetilde{R}_i^P), \Pi_{i=k}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right) \\ = {} & \left((1+r)^{n-k+1} + \frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left(1-\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right) \\ & \times\left((1+r)^{n-k+1} - \mathbb{E}\left(\Pi_{i=k}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right)\right)\end{aligned}$$

and

$$\mathbb{VAR}\left(\Pi_{i=k}^{n}(1+\widetilde{R}_i^P)|\mathcal{H}_{k-1}\right) \tag{2.9}$$

$$= \left((1+r)^{n-k+1} + \frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\right)^2 \mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right] \times \left(1 - \mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right]\right)$$

for all values of $k = 1, 2, \ldots n$ *and all trajectories such that* $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P) > 0$.

Now we are in the position to propose a solution to the classical multiperiod Markowitz problem in the case where the investor as a portfolio consisting of one riskless asset with constant rate of return and one risky asset.

THEOREM 2.8 *Given is some constant* $c > 0$*, the overall target rate of return. Assume that the portfolio weights* ω_k*,* $k = 1, ..., n$ *appearing in* (2.3) *are such that, under the conventions detailed immediately prior to the statement of lemma 2.2, there holds* $\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right] = 1 + c \geq (1+r)^n$*. The specific choice of weight defined by setting*

$$\lambda_n = 2\left[\frac{(1+r)^n - (1+c)}{\sum_{i=0}^{n-1}\mathbb{E}(\tau_i)} - (1+r)^n\right] \tag{2.10}$$

in (2.3) *minimizes the variance* $\mathbb{VAR}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]$*, on the set of all trajectories such that* $\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P) > 0$*, its minimal value being given by*

$$\mathbb{VAR}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right] = ((1+r)^n - (1+c))^2\left[\frac{1}{\sum_{i=0}^{n-1}\mathbb{E}(\tau_i)} - 1\right]. \tag{2.11}$$

COROLLARY 2.9 *In theorem* (2.8)*, if we enlarge the family of portfolios to all those satisfying* $\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right] = 1+c \geq (1+r)^n$ *then the optimal solution remains the same. (This follows immediately because the variance is an increasing function of c)*

The following shows that a solution to the multiperiod mean-variance problem gives rise to a natural extension of the classical capital asset pricing model in a multiperiod setting.

COROLLARY 2.10 *Let* P *be a portfolio with weights* ω_k*,* $k = 1, ..., n$ *satisfying the conditions of proposition 2.8 then for every other portfolio* Q *we have*

$$\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i^Q)\right] - (1+r)^n = \beta_n\left(\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i^P)\right] - (1+r)^n\right) \tag{2.12}$$

where

$$\beta_n = \frac{\mathbb{COV}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P), \Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right)}{\mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right)}. \tag{2.13}$$

REMARK 2.11 *When $n = 1$ one recovers the classical (single period) CAPM. See, for example Pliska [18].*

3 EXAMPLES

In this section, we shall exhibit several rich classes of models for the excess rate of return $(R_n - r)$ of a risky asset at time n. These models were selected in order to satisfy certain basic criteria, namely simplicity of use, computational efficiency and interpretability. All of them offer a simplified symbolic representation for the optimal values of the parameters τ_k, ω_k and λ_k defining the solution to the multiperiod mean-variance portfolio, given by equations (2.2), (2.3) and (2.10). Many will be seen to afford computational implementations with tractability throughout the calculation process, thereby ensuring accurate results in real time, even when parameter estimation is required. Lastly, the models will allow for interpretation and insight.

EXAMPLE 3.1 Our first example is a very general class of models driven by *independent multiplicative market impulses* (hereafter called the IMMI class). To describe this class, we first need a sequence of independent random variables $\{\xi_k : k \geq 1\}$, where ξ_n represents the random fluctuations (the noise) of the relevant part of the market at time n. All we require of the sequence $\{\xi_k : k \geq 1\}$ is that it should be adapted to the filtration $\{\mathcal{H}_k : k \geq 1\}$, which just means that ξ_k is $\mathcal{H}_k$-measurable for each $k \geq 1$; and that it should be independent of the whole past of the market, in other words, that ξ_k is not only independent of $\xi_1, \xi_2, \ldots, \xi_{k-1}$ but of the whole of $\mathcal{H}_{k-1}$.

More precisely, we assume that $(R_n - r)$ is given by some $\mathcal{H}_{n-1}$-measurable real-valued random variable Ψ_{n-1} which is perturbed by real-valued market impulse $\Phi_n(\xi_n)$, in the following multiplicative form : for every $n \geq 1$, we have

$$R_n - r = \Phi_n(\xi_n) \cdot \Psi_{n-1}. \tag{3.1}$$

Here Φ_n denotes some real-valued measurable mapping, arbitrary for now but key to the nature of the market fluctuations for the risky asset under consideration. Whereas the choice for Ψ_{n-1} is entirely up to you — pick your favorite predictive model for the average behavior of the excess rate for this asset! For instance, any measurable function giving rise to random variable Ψ_{n-1} in the guise $\Psi_{n-1} = \Psi_{n-1}(\xi_1, \xi_2, \ldots, \xi_{n-1}, R_1, R_2, \ldots, R_{n-1})$ will do. Specific examples are given below.

In fact, we shall see that Ψ_{n-1} is completely irrelevant to the determination of two of the families of parameters ($\{\tau_i\}$ and $\{\lambda_i\}$)— its influence is felt only in the determination of the third family (the weights ω_i themselves), as one sees clearly in formula (2.10) for λ_i and in the forthcoming simplified formula (3.4) for τ_i. This observation will be important when discussing estimation procedures in practical settings — read on to the next example for details.

This IMMI class is characterized by the fact that the noise source acts by dilating or compressing the signal, rather than translating it, as do additive models, thereby ensuring (by way of a scaling effect) that the fluctuations of the excess rate around its mean value are heteroscedastic in all but the most trivial special cases.

Another important observation to make about the IMMI class as a whole, is that combining its defining equation (3.1) together with our assumptions on both market noise and predictive model, implies

$$\mathbb{E}\left[R_n - r|\mathcal{H}_{n-1}\right] = \mathbb{E}\left(\Phi_n\left(\xi_n\right)\right) \cdot \Psi_{n-1}. \tag{3.2}$$

This simple representation for the one-step ahead average value of the excess rate, allows us to see at once that some simple restrictions on said noise and model will ensure some very good features indeed. When a bull market is expected for the risky asset, the sequence $\{R_n - r\}$ should form a submartingale and a clearly sufficient condition for this to happen is for both sequences $\{\mathbb{E}\left(\Phi_n\left(\xi_n\right)\right)\}$ and $\{\Psi_{n-1}\}$ to remain strictly positive at all times. When a bear market is in the cards, $\{R_n - r\}$ should form a supermartingale instead and for this, it is sufficient for the sequence $\{\mathbb{E}\left(\Phi_n\left(\xi_n\right)\right)\}$ to remain strictly positive at all times while the sequence $\{\Psi_{n-1}\}$ remains strictly negative. Note that good modelling entails strict positivity of the sequence of average noise $\{\mathbb{E}\left(\Phi_n\left(\xi_n\right)\right)\}$ in all circumstances, since otherwise at some time n our predictive model Ψ_{n-1} would have a (highly undesirable) mean of opposite sign from that of its intended target $R_n - r$.

Let us begin with the expression for τ_i in (2.2). Independence of the impulses turns all these random parameters into constants and reduces their successive values (for $i = n-1, n-2, \ldots, 2, 1, 0$) to

$$\tau_{n-1} = \frac{\mathbb{E}^2\left[(R_n - r)|\mathcal{H}_{n-1}\right]}{\mathbb{E}\left[(R_n - r)^2|\mathcal{H}_{n-1}\right]} = \frac{\mathbb{E}^2\Phi_n(\xi_n)}{\mathbb{E}\Phi_n^2(\xi_n)}, \tag{3.3}$$

$$\tau_i = \left(1 - \sum_{j=i+1}^{n-1}\tau_j\right)\frac{\mathbb{E}^2\Phi_{i+1}(\xi_{i+1})}{\mathbb{E}\Phi_{i+1}^2(\xi_{i+1})}, \tag{3.4}$$

$$\tau_0 = \left(1 - \sum_{j=1}^{n-1}\tau_j\right)\frac{\mathbb{E}^2\Phi_1(\xi_1)}{\mathbb{E}\Phi_1^2(\xi_1)}. \tag{3.5}$$

The expectation in (2.10) for λ_n can be removed for the IMMI class; the optimal weight ω_k associated to the risky asset in (2.3), now simplifies to

$$(3.6)\qquad \omega_k = -\left((1+r) + \frac{\lambda_n}{2(1+r)^{n-k}\Pi_{i=1}^{k-1}(1+\widetilde{R}_i)}\right)\frac{\mathbb{E}\Phi_k(\xi_k)}{\Psi_{k-1}\mathbb{E}\Phi_k^2(\xi_k)}.$$

The reader will note that while some simplifications have been obtained for the defining parameters in (2.2), (2.3) and (2.10) by working with the general IMMI class, the number of parameters to be estimated remains too high for practical purposes. The SIMMI subclass defined next will correct this shortcoming.

Example 3.2 When the driving noise in the IMMI class is assumed to be of the form $\{\Phi(\xi_k) : k \geq 1\}$, with a single function Φ and a sequence $\{\xi_k : k \geq 1\}$ which is now assumed to be not only independent, but identically distributed as well, we get what will be called the *stationary IMMI* class (for short, the SIMMI class). (Beware! The sequence of excess rates of return $\{R_k - r : k \geq 1\}$ will not in general form a stationary stochastic process, only the sequence of market impulses $\{\Phi(\xi_k) : k \geq 1\}$ will exhibit this property, since it is built into it.) Because of these restrictions, the parameters $\{\tau_i : i = 0, 1, \ldots, n-1\}$ in (2.2) now form the first few terms of a geometric progression : if we denote

$$(3.7)\qquad \rho = \tau_{n-1} = \frac{\mathbb{E}^2\Phi(\xi_1)}{\mathbb{E}\Phi^2(\xi_1)},$$

then clearly we have $\tau_{n-2} = \rho(1-\rho)$ and more generally

$$(3.8)\qquad \tau_i = \rho(1-\rho)^{n-i-1} \text{ for } i = 0, 1, 2, \ldots, n-1.$$

Furthermore, expression (2.10) becomes

$$(3.9)\qquad \lambda_n = 2\left[\frac{(1+r)^n - (1+c)}{1-(1-\rho)^n} - (1+r)^n\right].$$

It follows that, with a selection from the SIMMI class, at the preceding reinvestment time $k-1$, both parameters λ_k and τ_k are completely known functions of the single unknown parameter $\rho \in [0,1]$, which must be estimated by some well chosen function $\widehat{\rho}_n$ of the data $\{c, r, R_1, \ldots, R_{k-1}\}$ - for instance, one might choose the asymptotically unbiased moment estimator given by

$$\widehat{\rho}_n = \frac{\left[\sum_{i=1}^n \Phi(\xi_i)\right]^2}{n\sum_{i=1}^n \Phi^2(\xi_i)},$$

whereas optimal portfolio weight ω_k can now be written in the form

$$(3.10)\qquad \omega_k = -\left((1+r) + \frac{\lambda_n}{2(1+r)^{n-k}\Pi_{i=1}^{k-1}(1+\widetilde{R}_i)}\right)\frac{\rho}{\Psi_{k-1}\mathbb{E}\Phi(\xi_1)},$$

a known function of ρ, $\mathbb{E}\Phi\left(\xi_1\right)$ and Ψ_{k-1}. Therefore, computation of the optimal weights can be effected explicitly with the estimation of only two parameters (ρ and $\mathbb{E}\Phi\left(\xi_1\right)$), once the true market signals $\{\Psi_j : j < k\}$ for all preceding times, have been extracted from the noisy data $\{R_j : j < k\}$. More on this in the special cases below.

Another nicety about the SIMMI class is that heteroscedasticity remains built into it, just as in the larger IMMI class, even though the market impulse is clearly homoscedastic in the SIMMI class. The reason for this is the multiplicative nature of (3.1), which renders the variance of R_n non constant in all but the most trivial examples (the Cox-Rubinstein model being a case in point, as we shall see shortly).

We are now ready to investigate a number of members of the SIMMI class of models, which possess all three desired properties of a good model stated at the beginning of the present section.

Example 3.3 The *Cox-Rubinstein* model turns out to be in the SIMMI class. Just take $\Phi(\xi_k) = \mu - r + \sigma\xi_k$ (with $\mathbb{E}\xi_k = 0$ and $\mathbb{E}\xi_k^2 = 1$) and Ψ_k identically equal to one, for all values of $k \geq 0$. The Cox-Rubinstein model is thus written $R_n - r = \mu - r + \sigma\xi_n$, the excess rates of return itself now being independent and identically distributed from one time point to another. The optimal solution is still given by (3.6), with (3.9), (3.8) and (3.7), but now simplifies to

$$\omega_k = -\left((1+r) + \frac{\lambda_n}{2\,(1+r)^{n-k}\,\Pi_{i=1}^{k-1}\left(1+\widetilde{R}_i\right)}\right)\frac{\mu - r}{\left(\sigma^2 + (\mu - r)^2\right)} \tag{3.11}$$

with $\rho = (\mu - r)^2 / (\sigma^2 + (\mu - r)^2)$.

Estimation of the parameters here reduces to that of the first and second moments μ and σ^2 since Ψ_k is known. Unbiased estimators are readily obtained : just take the sample mean and variance for the observations R_n up to the present time. This example also brings forth yet another nice property of the SIMMI class, namely that it also includes linear and non multiplicative models through the incorporation of some market modelling into the noise, thus making the class even richer.

Example 3.4 It is very satisfying to notice that all classical and non-centered ARCH(p,q), MARCH(p,q), as well as those GARCH(p,q) models such that ξ_n form independent identically distributed random variables, also belong to the SIMMI class.

Indeed, following Guégan [7] (chapter 5), GARCH(p,q) models can be written in the form (3.1) with each market impulse ξ_n obeying a normal distribution $N(\mu - r, \sigma^2)$ (here $\Phi(\xi) = \xi$) and market signals of the form $\Psi_{n-1} = \sqrt{h_{n-1}}$ where the structure

$$h_{n-1} = \alpha_0 + \sum\nolimits_{i=1}^{q} \alpha_i (R_{n-i} - r)^2 + \sum\nolimits_{j=1}^{p} \beta_j h_{n-j} \tag{3.12}$$

requires $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $q > 0$ and $p \geq 0$.

Note that ARCH(p,q) models are simply those for which $\beta_j = 0$.

Meanwhile MARCH(p,q) models are exactly of the form (3.1) when ξ_n has a normal distribution $N(\mu - r, 1)$ (here again $\Phi(\xi) = \xi$) and $\Psi_{n-1} = \sqrt{h_{n-1}}$ is now given by

$$h_{n-1} = \sigma \cdot \prod_{i=1}^{q} (\xi_{n-i})^{2\alpha_i} \cdot \prod_{j=1}^{p} (R_{n-j} - r)^{2\beta_i} \tag{3.13}$$

provided the roots of polynomials $1 - \alpha_1 x - \ldots - \alpha_q x^q$ and $1 - \beta_1 x - \ldots - \beta_p x^p$ are all larger than 1 and distinct, in order to preclude explosion in finite time. The classical GARCH(p,q) and MARCH(p,q) models correspond to the case where $\tau_0 = \mu - r = 0$. The condition $\mathbb{E}(\tau_0) > 0$ in lemma 2.3 is equivalent here to the case $\mu > r$ of noncentral GARCH(p,q) and MARCH(p,q) processes.

It is important to note here that all MARCH(p,q) models are stationary in the wide sense (the autocorrelations are null since the multiplicative noises are centered and independent of the past), which renders their use with excess rate data sometimes unrealistic. Those GARCH(p,q) models suffering the same fate are identified in Bollerslev [1].

In both cases, the statistical estimation of all these parameters has been the subject of much research and the reader will find explicit solutions proposed in Guégan [7] (chapter 5) for the classical GARCH(p,q) model and Brockwell and Davis [2] (chapter 8) for the classical MARCH(p,q) model, once it is noticed (for this last collection) that $log(R_n - r)^2$ is actually a classical ARMA(p,q) model with nongaussian noise as soon as $R_n - r$ is assumed to follow a MARCH(p,q) model.

Now the following examples will allow us to compare, by mean of simulation, the numerical precision as well as the real time calculation of the optimal solution for two widely used models in mathematical finance.

Example 3.5 Let $\{S_j, j = 1 \ldots k\}$ the set of all possible states, X^0 a given random variable with distribution $\mathbb{P}\left(X^0 = S_j\right) = p_j^0$, $\{X_i^n, i = 1 \ldots k\}$ independent random variables with stationary distribution $\mathbb{P}(X_i^n = S_j) = p_{ij}$.

Then $\{R_j - r, j = 1 \ldots n\}$ follows a stationary Markov chain model with possible states $\{S_j, j = 1 \ldots k\}$ which means that $\{R_j - r, j = 1 \ldots n\}$ satisfies

$$\begin{aligned} R_0 - r &= X^0 \\ R_n - r &= \sum_{i=1}^{k} X_i^n I_i (R_{n-1} - r) \end{aligned}$$

where the function $I_i(u)$ is equal to 1 if $u = S_i$ and 0 otherwise.

By setting

$$\begin{aligned} \Psi_{n-1} &= (\Psi_{nj})_{j=1}^{k} \text{ with } \Psi_{nj} = I_j (R_{n-1} - r) \\ \Phi_n &= (\Phi_{nj})_{j=1}^{k} \text{ with } \Phi_{nj} = X_j^n \end{aligned}$$

we obtain

$$R_n - r = \Phi_n \bullet \Psi_{n-1}$$

where $\bullet$ denotes the scalar product in $\mathbb{R}^k$.

This lead us to notice that this model does not belong to the IMMI class but might well belong to an even larger class, namely the vectorial version of the IMMI class. In fact the model is a superposition of SIMMI models.

Let c be the desired expected global rate of return of the investor's portfolio, r the periodical interest rate, n the number of periods, k the number of states, S the state matrix, P the transition matrix $[p_{i,l}]_{k\times k}$, P_0 the initial distribution matrix $[p_i]_{i=1}^k$. For instance let $r = \frac{0.05}{365}$ (5% annual interest rate compounded daily), $c = \left(1 + \frac{0.06}{365}\right)^n - 1$ (6% annual growth rate compounded daily) , $k = 2$,

$$\begin{aligned} S &= \left[\begin{array}{cc} \frac{0.17}{365} & -\frac{0.11}{365} \end{array}\right], \\ P_0 &= \left[\begin{array}{cc} \frac{4}{7} & \frac{3}{7} \end{array}\right], \\ P &= \left[\begin{array}{cc} 0.7 & 0.3 \\ 0.4 & 0.6 \end{array}\right]. \end{aligned}$$

First, notice that for every $j = 1 \ldots n$,

$$\begin{aligned} \mathbb{E}\left(R_j - r | \mathcal{H}_{j-1}\right) &= \begin{cases} \frac{0.086}{365} & \text{if} \quad R_{j-1} - r = \frac{0.17}{365} \\ \frac{0.002}{365} & \text{if} \quad R_{j-1} - r = -\frac{0.11}{365} \end{cases} \\ \mathbb{VAR}\left(R_j - r | \mathcal{H}_{j-1}\right) &= \begin{cases} \frac{0.016464}{365^2} & \text{if} \quad R_{j-1} - r = \frac{0.17}{365} \\ \frac{0.018816}{365^2} & \text{if} \quad R_{j-1} - r = -\frac{0.11}{365} \end{cases} \end{aligned}$$

thus $\mathbb{E}\left(R_j - r\right) = \frac{0.05}{365}$ and $\mathbb{VAR}\left(R_j - r\right) = \frac{0.0192}{365^2}$.

Following each time the optimal portfolio strategy, the tables shown below gives the main characteristics obtained from 200 simulations using the Maple routine given in Appendix A of Watier [25] for each of the $n = 30$, $n = 90$, $n = 180$ horizons.

$n = 30$	Values
Desired global rate of return c	0.004943279454
Average global rate of return	0.004937887242
Variance of the global rate of return	$0.101021453610 \times 10^{-7}$
Average annual rate of return (compounded daily)	0.059934706561
Average calculation time of the strategy (seconds)	2.79

$n = 90$	Values
Desired global rate of return c	0.014903267192
Average global rate of return	0.014903294158
Variance of the global rate of return	$0.371719591910 \times 10^{-15}$
Average annual rate of return (compounded daily)	0.060000107774
Average calculation time of the strategy (seconds)	8.95

$n = 180$	Values
Desired global rate of return c	0.030028641757
Average global rate of return	0.030028641758
Variance of the global rate of return	$0.135582624510 \times 10^{-27}$
Average annual rate of return (compounded daily)	0.060000000001
Average calculation time of the strategy (seconds)	20.78

Example 3.6 Let$\{S_j, j = 1 \dots k\}$ the set of all possible states, $\{X_i^n, i = 1 \dots k\}$ independent random variables with stationary distribution $\mathbb{P}(X_i^n = S_j) = p_{ij}$ independent of i.

Then $\{R_j - r, j = 1 \dots n\}$ follow a multinomial tree model with possible states $\{S_j, j = 1 \dots k\}$ and following the previous example, we obtain

$$\begin{aligned} R_n - r &= X^n \\ &= \Phi_n \bullet 1. \end{aligned}$$

Consequently this model belongs to the SIMMI class where

$$\mathbb{P}(R_j - r = S_i | \mathcal{H}_{j-1}) = \mathbb{P}(R_j - r = S_i) = p_i.$$

The very popular multinomial tree model is just a special case of the Markov chain models of example 3.5. We therefore use the same notation and numerical values for the sake of comparisons, with the exception of the transition matrix, here equal to that prescribed by the stationary distribution from example 3.5, namely

$$P = \begin{bmatrix} \frac{4}{7} & \frac{3}{7} \\ \frac{4}{7} & \frac{3}{7} \end{bmatrix}.$$

First, notice that for every $j = 1 \dots n$,

$$\begin{aligned} \mathbb{E}(R_j - r | \mathcal{H}_{j-1}) &= \mathbb{E}(R_j - r) = \frac{0.05}{365} \\ \mathbb{VAR}(R_j - r | \mathcal{H}_{j-1}) &= \mathbb{VAR}(R_j - r) = \frac{0.0192}{365^2} \end{aligned}$$

thus $\mathbb{E}(R_j - r)$ and $\mathbb{VAR}(R_j - r)$ are the same as the preceding markovian model.

The tables shown below gives the main characteristics obtained from 1000 simulations using the Maple routine given in Appendix B of Watier [25] for each of the $n = 30$, $n = 90$, $n = 180$ horizons.

$n = 30$	Values
Desired global rate of return c	0.004943279454
Average global rate of return	0.004944582121
Variance of the global rate of return	$0.599072752710 \times 10^{-8}$
Average annual rate of return (compounded daily)	0.060015773736
Average calculation time of the strategy (seconds)	0.14

$n = 90$	Values
Desired global rate of return c	0.014903267192
Average global rate of return	0.014903288190
Variance of the global rate of return	$0.382160534110 \times 10^{-13}$
Average annual rate of return (compounded daily)	0.060000083921
Average calculation time of the strategy (seconds)	0.41

$n = 180$	Values
Desired global rate of return c	0.030028641757
Average global rate of return	0.030028641758
Variance of the global rate of return	$0.311035740510 \times 10^{-22}$
Average annual rate of return (compounded daily)	0.060000000002
Average calculation time of the strategy (seconds)	0.99

We notice that both models give similar numerical results but the average calculation time of a strategy is about 20 times superior in the two state markovian model compared to the binomial model.

4 PROOFS

4.1 Proof of lemma 2.2

We will use a backward induction on k. The property is easily satisfied in the base case (for $k = n - 1$) that is

$$0 \leq \tau_{n-1} = \frac{\mathbb{E}^2\left[(R_n - r)|\mathcal{H}_{n-1}\right]}{\mathbb{E}\left[(R_n - r)^2|\mathcal{H}_{n-1}\right]} \leq 1. \tag{4.1}$$

Suppose that the inequalities are satisfies for $k = l$ that is

$$0 \leq \tau_l \leq \mathbb{E}\left[1 - \sum\nolimits_{i=l+1}^{n-1} \tau_i|\mathcal{H}_l\right] \leq 1 \tag{4.2}$$

then to show the first inequality $0 \leq \tau_{l-1}$ we only need to prove that the denominator of $\tau_{l-1}$is positive. From the second inequality in (4.2) we obtain

$$\begin{aligned}
\tau_l &\leq \mathbb{E}\left[1 - \sum\nolimits_{i=l+1}^{n-1} \tau_i|\mathcal{H}_l\right] \\
\Longrightarrow 0 &\leq \mathbb{E}\left[1 - \sum\nolimits_{i=l}^{n-1} \tau_l|\mathcal{H}_l\right] \\
\Longrightarrow 0 &\leq \mathbb{E}\left[1 - \sum\nolimits_{i=l}^{n-1} \tau_l|\mathcal{H}_l\right](R_l - r)^2 \\
\Longrightarrow 0 &\leq \mathbb{E}\left[\mathbb{E}\left[1 - \sum\nolimits_{i=l}^{n-1} \tau_l|\mathcal{H}_l\right](R_l - r)^2|\mathcal{H}_{l-1}\right] \\
\Longrightarrow 0 &\leq \mathbb{E}\left[\left(1 - \sum\nolimits_{i=l}^{n-1} \tau_l\right)(R_l - r)^2|\mathcal{H}_{l-1}\right]
\end{aligned} \tag{4.3}$$

now we prove the second inequality $\tau_{l-1} \leq \mathbb{E}\left[1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right]$, from (4.3) we may write τ_{l-1} as follows

$$\begin{aligned}
\tau_{l-1} &= \frac{\mathbb{E}^2\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)|\mathcal{H}_{l-1}\right]}{\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]} \\
&= \frac{\mathbb{E}^2\left[\mathbb{E}\left(1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right)(R_l-r)|\mathcal{H}_{l-1}\right]}{\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]}
\end{aligned}$$

$$= \frac{\mathbb{E}^2\left[\left(\mathbb{E}\left(1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right)\right)^{1/2}\left(\mathbb{E}\left(1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right)\right)^{1/2}(R_l-r)|\mathcal{H}_{l-1}\right]}{\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]}.$$

Applying the Cauchy-Schwarz inequality to the numerator we have

$$\begin{aligned}
\tau_{l-1} &\leq \frac{\mathbb{E}\left[\mathbb{E}\left(1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right]\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]}{\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]} \\
&= \frac{\mathbb{E}\left[1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]}{\mathbb{E}\left[\left(1-\sum_{i=l}^{n-1}\tau_i\right)(R_l-r)^2|\mathcal{H}_{l-1}\right]} \\
&= \mathbb{E}\left[1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right].
\end{aligned}$$

Finally we show the third inequality $\mathbb{E}\left[1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right] \leq 1$, now using both the first and third inequality in (4.2), we obtain

$$\begin{aligned}
\mathbb{E}\left[1-\sum_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right] &= \mathbb{E}\left[\left(\mathbb{E}\left(1-\sum_{i=l+1}^{n-1}\tau_i|\mathcal{H}_l\right)-\tau_l\right)|\mathcal{H}_{l-1}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left(1-\sum_{i=l+1}^{n-1}\tau_i|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right]-\mathbb{E}\left[\tau_l|\mathcal{H}_{l-1}\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left(1-\sum_{i=l+1}^{n-1}\tau_i|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right] \\
&\leq 1.
\end{aligned}$$

4.2 Proof of lemma 2.3

The first condition immediately yields the well posedness of formula (2.10) for each and every $\{\lambda_n : n = 1, 2, ...\}$. The second condition ensures that the first denominator in (2.3) for each and every $\{\omega_k : k = 1, ..., n\}$ is non null. Since formulas (2.2) and (2.3) will thus both be well-defined once we have established that a null denominator in (2.2) implies a null numerator in both (2.2) and (2.3), this is all we have left to do. Applying the Cauchy-Schwarz inequality to a split form of the numerator of τ_i in (2.2) we get

$$\begin{aligned} 0 &\leq \mathbb{E}^2\left[\left(1-\sum_{j=i+1}^{n-1}\tau_j\right)(R_{i+1}-r)\,|\mathcal{H}_i\right] \\ &\leq \mathbb{E}\left[\left(1-\sum_{j=i+1}^{n-1}\tau_j\right)|\mathcal{H}_i\right]\cdot\mathbb{E}\left[\left(1-\sum_{j=i+1}^{n-1}\tau_j\right)(R_{i+1}-r)^2\,|\mathcal{H}_i\right] \end{aligned}$$

using lemma 2.2 to establish the nonnegativity of the terms under the square roots. Hence if the RHS is null so is the LHS.

4.3 Proof of lemma 2.4

By backward induction. The property is easily satisfied for $k=n$ using formulae (2.2) and (2.3). Notice that, by the tower property

$$\mathbb{E}\left[\Pi_{i=k-1}^{n}(1+\widetilde{R}_i)|\mathcal{H}_{k-2}\right]=\mathbb{E}\left[(1+\widetilde{R}_{k-1})\mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i)|\mathcal{H}_{k-1}\right]|\mathcal{H}_{k-2}\right].$$

Now suppose the property is verified for $k=l$, with $\omega_l,...,\omega_n$ as defined by (2.3), by the induction hypothesis, we obtain

$$\begin{aligned} &\mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i)|\mathcal{H}_{l-2}\right] \\ =\ &(1+r)^{n-l+1}\,\mathbb{E}\left((1+\widetilde{R}_{l-1})\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right) \\ &-\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)}\mathbb{E}\left(\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\ =\ &(1+r)^{n-l+2}\,\mathbb{E}\left(1-\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\ &+(1+r)^{n-l+1}\,\omega_{l-1}\mathbb{E}\left(\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)(R_{l-1}-r)\,|\mathcal{H}_{l-2}\right) \\ &-\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)}\mathbb{E}\left(\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right). \end{aligned}$$

Substituting the value of ω_{l-1} in the last equation, we deduce

$$\begin{aligned} \mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i)|\mathcal{H}_{l-2}\right] &= (1+r)^{n-l+2}\,\mathbb{E}\left(1-\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\ &\quad-\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)}\mathbb{E}\left(\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right). \end{aligned}$$

4.4 Proof of lemma 2.5

By backward induction. The property is easily satisfied for $k=n$, just like for the base case in the previous proof . Notice that, by the tower property

$$
\begin{aligned}
& \mathbb{E}\left[\Pi_{i=k-1}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_{k-2}\right] \\
= \; & \mathbb{E}\left[(1+\widetilde{R}_{k-1})^2\mathbb{E}\left[\Pi_{i=k}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_{k-1}\right]|\mathcal{H}_{k-2}\right]
\end{aligned}
$$

Now suppose the property is verified for $k=l$, with $\omega_l,...,\omega_n$ as defined by (2.3), by the induction hypothesis, we obtain

$$
\begin{aligned}
& \mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_{l-2}\right] \\
= \; & (1+r)^{2(n-l+1)}\,\mathbb{E}\left((1+\widetilde{R}_{l-1})^2\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right) \\
& +\frac{\lambda_n^2}{4\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)^2}\mathbb{E}\left(\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\
= \; & (1+r)^{2(n-l+2)}\,\mathbb{E}\left(1-\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\
& +2\,(1+r)^{2(n-l+1)+1}\,\omega_{l-1}\mathbb{E}\left(\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)(R_{l-1}-r)\,|\mathcal{H}_{l-2}\right) \\
& +(1+r)^{2(n-l+1)}\,\omega_{l-1}^2\mathbb{E}\left(\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)(R_{l-1}-r)^2\,|\mathcal{H}_{l-2}\right) \\
& +\frac{\lambda_n^2}{4\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)^2}\mathbb{E}\left(\sum_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right).
\end{aligned}
$$

Substituting the value of ω_{l-1} in the last equation, we deduce

$$
\begin{aligned}
\mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_{l-2}\right] & =(1+r)^{2(n-l+2)}\,\mathbb{E}\left(1-\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right) \\
& +\frac{\lambda_n^2}{4\Pi_{i=1}^{l-2}(1+\widetilde{R}_i)^2}\mathbb{E}\left(\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right).
\end{aligned}
$$

4.5 Proof of lemma 2.6

Let γ_i be the $\mathcal{H}_{i-1}$ -measurable random variable corresponding to the weight of the arbitrary portfolio Q allocated to the risky asset just before time i. The proof involves once again a backward induction. However the base case $k=n$ here is nontrivial and we provide the details of the proof. Indeed there comes

$$
\begin{aligned}
& \mathbb{E}\left[(1+\widetilde{R}_n^P)(1+\widetilde{R}_n^Q)|\mathcal{H}_{n-1}\right] \\
= \; & \mathbb{E}\left[((1+r)+\omega_n\,(R_n-r))\,((1+r)+\gamma_n\,(R_n-r))\,|\mathcal{H}_{n-1}\right] \\
= \; & (1+r)^2+(1+r)\,\omega_n\mathbb{E}\left[R_n-r|\mathcal{H}_{n-1}\right]+(1+r)\,\gamma_n\mathbb{E}\left[R_n-r|\mathcal{H}_{n-1}\right] \\
& +\omega_n\gamma_n\mathbb{E}\left[(R_n-r)^2\,|\mathcal{H}_{n-1}\right].
\end{aligned}
$$

Substituting the value of ω_n in the last equation, we deduce

$$
\begin{aligned}
& \mathbb{E}\left[(1+\widetilde{R}_n^P)(1+\widetilde{R}_n^Q)|\mathcal{H}_{n-1}\right] \\
= \;& (1+r)^2-(1+r)\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\right)\tau_{n-1} \\
& +(1+r)\,\gamma_n\mathbb{E}\left[R_n-r|\mathcal{H}_{n-1}\right] \\
& -\gamma_n\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left[R_n-r|\mathcal{H}_{n-1}\right] \\
= \;& (1+r)^2-(1+r)\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\right)\tau_{n-1} \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\mathbb{E}\left[\gamma_n\left(R_n-r\right)|\mathcal{H}_{n-1}\right] \\
= \;& (1+r)^2\left(1-\tau_{n-1}\right)-(1+r)\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\tau_{n-1} \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\mathbb{E}\left[\left(1+\widetilde{R}_n^Q\right)-(1+r)\,|\mathcal{H}_{n-1}\right] \\
= \;& (1+r)\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\right)(1-\tau_{n-1}) \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{n-1}(1+\widetilde{R}_i^P)}\mathbb{E}\left[1+\widetilde{R}_n^Q|\mathcal{H}_{n-1}\right].
\end{aligned}
$$

For the remainder of the proof, we repeatedly need the tower property, namely for any $l=1,...,n$

$$
\begin{aligned}
& \mathbb{E}\left[\Pi_{i=l-1}^n(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-2}\right] \\
= \;& \mathbb{E}\left[(1+\widetilde{R}_{l-1}^P)(1+\widetilde{R}_{l-1}^Q)\mathbb{E}\left(\Pi_{i=l}^n(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-1}\right)|\mathcal{H}_{l-2}\right].
\end{aligned}
$$

Now suppose that property (2.7) is verified for $k=l$, with $\omega_l,...,\omega_n$ as defined by (2.3), by the induction hypothesis, we obtain

$$
\begin{aligned}
& \mathbb{E}\left[\Pi_{i=l-1}^n(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-2}\right] \\
= \;& \mathbb{E}\Bigg[(1+\widetilde{R}_{l-1}^P)(1+\widetilde{R}_{l-1}^Q)\left(1+r\right)^{2(n-l)+1} \\
& \times\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l}\Pi_{i=1}^{l-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left[1-\sum\nolimits_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-1}\right]\Bigg|\mathcal{H}_{l-2}\Bigg] \\
& -\mathbb{E}\left[\frac{\lambda_n}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i^P)}\mathbb{E}\left[(1+\widetilde{R}_{l-1}^P)(1+\widetilde{R}_{l-1}^Q)\Pi_{i=l}^n(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-1}\right]|\mathcal{H}_{l-2}\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& (1+r)^{2(n-l+1)}\,\mathbb{E}\left[(1+\widetilde{R}^P_{l-1})(1+\widetilde{R}^Q_{l-1})\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right] \\
& +(1+r)^{n-l+1}\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\mathbb{E}\left[(1+\widetilde{R}^Q_{l-1})\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right] \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}^Q_i)|\mathcal{H}_{l-2}\right].
\end{aligned}
$$

On the other hand, when writing $\widetilde{R}^Q_{l-1}-r=\gamma_{l-1}\left(R_{l-1}-r\right)$ we get (for the expectation in the first term after the last equality)

$$
\begin{aligned}
& \mathbb{E}\left[(1+\widetilde{R}^P_{l-1})(1+\widetilde{R}^Q_{l-1})\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right] \\
= \;& \mathbb{E}\left[\left((1+r)+\omega_{l-1}\left(R_{l-1}-r\right)\right)\left((1+r)+\gamma_{l-1}\left(R_{l-1}-r\right)\right)\right. \\
& \qquad \left.\times\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right].
\end{aligned}
$$

Substituting the value of ω_{l-1} in the last equation, we deduce

$$
\begin{aligned}
= \;& (1+r)^2\,\mathbb{E}\left[1-\sum\nolimits_{i=l-1}^{n-1}\tau_i|\mathcal{H}_{l-2}\right] \\
& -(1+r)\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\right)\tau_{l-2} \\
& +(1+r)\,\gamma_{l-1}\mathbb{E}\left[(R_{l-1}-r)\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right] \\
& -\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\right) \\
& \qquad \times\gamma_{l-1}\mathbb{E}\left[\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)\left(R_{l-1}-r\right)|\mathcal{H}_{l-2}\right] \\
= \;& (1+r)^2\,\mathbb{E}\left[1-\sum\nolimits_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right]-\frac{\lambda_n}{2(1+r)^{n-l}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\tau_{l-2} \\
& -\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)} \\
& \qquad \times\mathbb{E}\left[\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)\left(\left(1+\widetilde{R}^Q_{l-1}\right)-(1+r)\right)|\mathcal{H}_{l-2}\right] \\
= \;& (1+r)^2\,\mathbb{E}\left[1-\sum\nolimits_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right] \\
& +\frac{\lambda_n}{2(1+r)^{n-l}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\mathbb{E}\left[1-\sum\nolimits_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right] \\
& -\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\mathbb{E}\left[\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)\left(1+\widetilde{R}^Q_{l-1}\right)|\mathcal{H}_{l-2}\right] \\
= \;& (1+r)\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\right)\mathbb{E}\left[1-\sum\nolimits_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right] \\
& -\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}^P_i)}\mathbb{E}\left[\left(1-\sum\nolimits_{i=l-1}^{n-1}\tau_i\right)\left(1+\widetilde{R}^Q_{l-1}\right)|\mathcal{H}_{l-2}\right].
\end{aligned}
$$

Thus it ensues

$$
\begin{aligned}
& \mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-2}\right] \\
= \;& (1+r)^{2(n-l+1)+1}\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)}\right. \\
& \left.\times\mathbb{E}\left[1-\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right]\right. \\
& -(1+r)^{2(n-l+1)}\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)} \\
& \times\mathbb{E}\left[\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)\left(1+\widetilde{R}_{l-1}^Q\right)|\mathcal{H}_{l-2}\right] \\
& +(1+r)^{n-l+1}\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)}\mathbb{E}\left[(1+\widetilde{R}_{l-1}^Q)\left(1-\sum_{i=l-1}^{n-1}\tau_i\right)|\mathcal{H}_{l-2}\right] \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)}\mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-2}\right] \\
= \;& (1+r)^{2(n-l+1)+1}\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-l+1}\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)}\right. \\
& \left.\times\mathbb{E}\left[1-\sum_{i=l-2}^{n-1}\tau_i|\mathcal{H}_{l-2}\right]\right. \\
& -\frac{\lambda_n}{2\Pi_{i=1}^{l-2}(1+\widetilde{R}_i^P)}\mathbb{E}\left[\Pi_{i=l-1}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{l-2}\right].
\end{aligned}
$$

4.6 Proof of theorem 2.8

We minimize $\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]^2$ under the constraint $\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]=1+c$ and the result will follow. Let us denote L^2 the space of square-integrable random variables on $(\Omega,\mathcal{F},\mathbb{P})$. Let $F:\left(L^2\right)^n\times\mathbb{R}\to\mathbb{R}$ be defined by :

$$
F(\omega_1,...,\omega_n,\lambda_n)=\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]^2+\lambda_n\left(\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]-(1+c)\right).
$$

A quick inspection of the definition of functional F reveals the fact that, as the expectation of a quadratic form in parameters ω_i for every value of i, it is a continuous (Gâteaux) differential functional on its defining space of square integrable functions (see page 32 of Clarke [3]). Hence it is also strictly differentiable and Clarke's version of the Lagrange Multiplier Rule (his theorem 6.1.1 on page 228) is therefore applicable to our optimization problem. Hence its solution must satisfy for each $k=1,...,n$

$$
\begin{aligned}
0 &= \frac{\partial F(\omega_1,...,\omega_n,\lambda_n)}{\partial\omega_k} \\
&= \frac{\partial}{\partial\omega_k}\left[\mathbb{E}\left(\Pi_{i=1}^{n}(1+\tilde{R}_i)^2\right)+\lambda_n\left(\mathbb{E}\left(\Pi_{i=1}^{n}(1+\tilde{R}_i)\right)-(1+c)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbb{E}\left[\frac{\partial}{\partial\omega_k}\left(\left(\Pi_{i=1}^n(1+\tilde{R}_i)^2\right)+\lambda_n\left(\left(\Pi_{i=1}^n(1+\tilde{R}_i)\right)-(1+c)\right)\right)\right] \\
&= \mathbb{E}\left(\left(2\Pi_{i=1}^n(1+\tilde{R}_i)+\lambda_n\right)\frac{\partial}{\partial\omega_k}\left(\Pi_{i=1}^n(1+\tilde{R}_i)\right)\right) \\
&= \mathbb{E}\left(\left(2\Pi_{i=1}^n(1+\tilde{R}_i)+\lambda_n\right)\left(\Pi_{i=1,i\neq k}^n(1+\tilde{R}_i)\right)(R_k-r)\right).
\end{aligned}
$$

Candidates for the maximum must satisfy this critical point property. Here is the key idea to the proof. Notice that possible solutions to these equations are obtained when the conditional expectation is equal to zero, in which case one observes

$$
\begin{aligned}
&\mathbb{E}\left(\left(2\Pi_{i=1}^n(1+\tilde{R}_i)+\lambda_n\right)\left(\Pi_{i=1,i\neq k}^n(1+\tilde{R}_i)\right)(R_k-r)\,|\mathcal{H}_{k-1}\right) \\
= \;& 2\left(\Pi_{i=1}^{k-1}(1+\tilde{R}_i)^2\right)\mathbb{E}\left(\left(1+\tilde{R}_k\right)\Pi_{i=k+1}^n(1+\tilde{R}_i)^2\,(R_k-r)\,|\mathcal{H}_{k-1}\right) \\
&+\lambda_n\Pi_{i=1}^{k-1}(1+\tilde{R}_i)\mathbb{E}\left(\left(\Pi_{i=k+1}^n(1+\tilde{R}_i)\right)(R_k-r)\,|\mathcal{H}_{k-1}\right) \\
= \;& 2\left(\Pi_{i=1}^{k-1}(1+\tilde{R}_i)^2\right) \\
&\times\left[\mathbb{E}\left(((1+r)+\omega_k\,(R_k-r))\,\Pi_{i=k+1}^n(1+\tilde{R}_i)^2\,(R_k-r)\,|\mathcal{H}_{k-1}\right)\right] \\
&+\lambda_n\Pi_{i=1}^{k-1}(1+\tilde{R}_i)\mathbb{E}\left(\left(\Pi_{i=k+1}^n(1+\tilde{R}_i)\right)(R_k-r)\,|\mathcal{H}_{k-1}\right) \\
= \;& 0.
\end{aligned}
$$

By isolating ω_k, which is $\mathcal{H}_{k-1}$-measurable, in the last expression we obtain, unless $X_k = 0$ that is bankruptcy occurs,

$$
\begin{aligned}
\omega_k \;=\;& \frac{-2(1+r)\Pi_{i=1}^{k-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_k-r)\Pi_{i=k+1}^n(1+\tilde{R}_i)^2|\mathcal{H}_{k-1}\right]}{2\Pi_{i=1}^{k-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_k-r)^2\Pi_{i=k+1}^n(1+\tilde{R}_i)^2|\mathcal{H}_{k-1}\right]} \\
&-\frac{\lambda_n\mathbb{E}\left[(R_k-r)\Pi_{i=k+1}^n(1+\tilde{R}_i)|\mathcal{H}_{k-1}\right]}{2\Pi_{i=1}^{k-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_k-r)^2\Pi_{i=k+1}^n(1+\tilde{R}_i)^2|\mathcal{H}_{k-1}\right]}.
\end{aligned}
\tag{4.4}
$$

We will show by backward induction that these ω_k are of the form (2.3). For the base case $k = n$ it is immediate.

Next, suppose that equation (2.3) is verified for $k = l+1, l+2, ..., n$. According to lemmas 2.4 and 2.5, the numerator of (4.4), with $k = l$, can be expressed as

$$
\begin{aligned}
&-2(1+r)\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)\mathbb{E}\left(\Pi_{i=l+1}^n(1+\tilde{R}_i)^2|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right] \\
&-\lambda_n\mathbb{E}\left[(R_l-r)\mathbb{E}\left(\Pi_{i=l+1}^n(1+\tilde{R}_i)|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & -2(1+r)\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)\left((1+r)^{2(n-l)}\,\mathbb{E}\left[1-\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\right)|\mathcal{H}_{l-1}\right] \\
& -2(1+r)\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)\frac{\lambda_n^2}{4\Pi_{i=1}^{l}(1+\widetilde{R}_i)^2}\mathbb{E}\left[\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]|\mathcal{H}_{l-1}\right] \\
& -\lambda_n\mathbb{E}\left[(R_l-r)\left((1+r)^{n-l}\,\mathbb{E}\left[1-\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\right)|\mathcal{H}_{l-1}\right] \\
& +\lambda_n\mathbb{E}\left[(R_l-r)\left(\frac{\lambda_n}{2\Pi_{i=1}^{l}(1+\widetilde{R}_i)}\mathbb{E}\left[\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\right)|\mathcal{H}_{l-1}\right] \\
= \quad & -2(1+r)\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\,\mathbb{E}\left[(R_l-r)\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right] \\
& -\lambda_n\,(1+r)^{n-l}\,\mathbb{E}\left[(R_l-r)\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right] \\
& -(1+r)\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\frac{(R_l-r)}{(1+\widetilde{R}_l)^2}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right] \\
& +\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\frac{(R_l-r)}{(1+\widetilde{R}_l)}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right] \\
= \quad & -2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)\,(1+r)^{n-l}}\right) \\
& \qquad\qquad\qquad\qquad \times\mathbb{E}\left[(R_l-r)\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right] \\
& +\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\omega_l\mathbb{E}\left[\frac{(R_l-r)^2}{(1+\widetilde{R}_l)^2}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right]\;;
\end{aligned}
$$

and the denominator as

$$
\begin{aligned}
& 2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)^2\mathbb{E}\left(\Pi_{i=l+1}^{n}(1+\tilde{R}_i)^2|\mathcal{H}_l\right)|\mathcal{H}_{l-1}\right] \\
= \quad & 2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)^2\left((1+r)^{2(n-l)}\,\mathbb{E}\left[1-\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\right)|\mathcal{H}_{l-1}\right] \\
& +2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\mathbb{E}\left[(R_l-r)^2\left(\frac{\lambda_n^2}{4\Pi_{i=1}^{l}(1+\widetilde{R}_i)^2}\mathbb{E}\left[\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]\right)|\mathcal{H}_{l-1}\right] \\
= \quad & 2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\,\mathbb{E}\left[(R_l-r)^2\mathbb{E}\left[1-\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]|\mathcal{H}_{l-1}\right] \\
& +\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\frac{(R_l-r)^2}{(1+\widetilde{R}_l)^2}\mathbb{E}\left[\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_l\right]|\mathcal{H}_{l-1}\right] \\
= \quad & 2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\,\mathbb{E}\left[(R_l-r)^2\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right] \\
& +\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\frac{(R_l-r)^2}{(1+\widetilde{R}_l)^2}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right].
\end{aligned}
$$

Thus we get by substituting these formulas for the numerator and denominator into (4.4) :

$$
\begin{aligned}
& \omega_l \left(2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\,\mathbb{E}\left[(R_l-r)^2\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right]\right) \\
+ & \;\omega_l\left(\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\mathbb{E}\left[\frac{(R_l-r)^2}{(1+\widetilde{R}_l)^2}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right]\right) \\
= & \; -2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)\,(1+r)^{n-l}}\right) \\
& \qquad \times\mathbb{E}\left[(R_l-r)\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right] \\
& +\frac{\lambda_n^2}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)}\omega_l\mathbb{E}\left[\frac{(R_l-r)^2}{(1+\widetilde{R}_l)^2}\sum\nolimits_{i=l}^{n-1}\tau_i|\mathcal{H}_{l-1}\right]
\end{aligned}
$$

$$
\begin{aligned}
\Longrightarrow & \quad \omega_l \left(2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\,\mathbb{E}\left[(R_l-r)^2\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right]\right) \\
= & \quad -2\Pi_{i=1}^{l-1}(1+\tilde{R}_i)\,(1+r)^{2(n-l)}\left((1+r)+\frac{\lambda_n}{2\Pi_{i=1}^{l-1}(1+\widetilde{R}_i)\,(1+r)^{n-l}}\right) \\
& \qquad \times\mathbb{E}\left[(R_l-r)\left(1-\sum\nolimits_{i=l}^{n-1}\tau_i\right)|\mathcal{H}_{l-1}\right]
\end{aligned}
$$

which is exactly (4.4) with $k=l$. The proof by induction is now complete.

The vector $\{\omega_k\}$ is then a critical point of the variational problem ; let us show that this vector is indeed a global maximum.

According to the tower property,

$$
1+c=\mathbb{E}[\mathbb{E}\left(\Pi_{i=1}^{n}(1+\widetilde{R}_i)|\mathcal{H}_0\right)]
$$

now using lemma 2.4, the constraint leads to

$$
\begin{aligned}
1+c &= \mathbb{E}\left[(1+r)^n\,\mathbb{E}\left[1-\sum\nolimits_{i=0}^{n-1}\tau_i|\mathcal{H}_0\right]-\frac{\lambda_n}{2}\mathbb{E}\left[\sum\nolimits_{i=0}^{n-1}\tau_i|\mathcal{H}_0\right]\right] \\
&= (1+r)^n\mathbb{E}\left[1-\sum\nolimits_{i=0}^{n-1}\tau_i\right]-\frac{\lambda_n}{2}\mathbb{E}\left[\sum\nolimits_{i=0}^{n-1}\tau_i\right].
\end{aligned}
$$

Isolating λ_n we obtain (2.10).

Furthermore for these weights ω_k we have, according to the tower property

$$
\begin{aligned}
& \mathbb{VAR}\left(\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right) \\
= & \;\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]^2-\left(\mathbb{E}\left[\Pi_{i=1}^{n}(1+\widetilde{R}_i)\right]\right)^2 \\
= & \;\mathbb{E}\left[\mathbb{E}\left(\Pi_{i=1}^{n}(1+\widetilde{R}_i)^2|\mathcal{H}_0\right)\right]-(1+c)^2.
\end{aligned}
$$

Now using lemma 2.5, we have

$$
\begin{aligned}
& \mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i)\right) \\
= \;& \mathbb{E}\left((1+r)^{2n}\,\mathbb{E}\left[1-\sum_{i=0}^{n-1}\tau_i|\mathcal{H}_0\right]+\frac{\lambda_n^2}{4}\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i|\mathcal{H}_0\right]\right)-(1+c)^2 \\
= \;& (1+r)^{2n}\,\mathbb{E}\left(1-\sum_{i=0}^{n-1}\tau_i\right)+\frac{\lambda_n^2}{4}\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)-(1+c)^2
\end{aligned}
$$

and substituting the value of λ_n we obtain

$$
\begin{aligned}
& \mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i)\right) \\
= \;& (1+r)^{2n}\,\mathbb{E}\left(1-\sum_{i=0}^{n-1}\tau_i\right) \\
& +\left[\frac{(1+r)^n-(1+c)}{\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right]}-(1+r)^n\right]^2\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)-(1+c)^2 \\
& = ((1+r)^n-(1+c))^2\left[\frac{1}{\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right]}-1\right].
\end{aligned}
$$

Finally we will show that the weights ω_k associated with the portfolio P of fixed mean $\mathbb{E}[\Pi_{i=1}^n(1+\widetilde{R}_i^P)]=1+c$ form the optimal solutions. Indeed we next show that

$$
\mathbb{COV}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P),\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right)=\mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right)
$$

holds for any other portfolio Q of same expected mean $\mathbb{E}[\Pi_{i=1}^n(1+\widetilde{R}_i^Q)]=1+c$.

This will imply by Cauchy-Schwartz that

$$
\mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right)\leq\sqrt{\mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right)\mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right)}.
$$

First notice that using the tower property

$$
\begin{aligned}
& \mathbb{COV}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P),\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right) \\
= \;& \mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right)-\mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right)\mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right) \\
= \;& \mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)\right)-(1+c)^2 \\
= \;& \mathbb{E}\left(\mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)(1+\widetilde{R}_i^Q)|\mathcal{H}_0\right)\right)-(1+c)^2\,.
\end{aligned}
$$

Now applying lemma 2.6

$$
\mathbb{COV}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P),\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right)
$$

$$
\begin{aligned}
&= \mathbb{E}\left((1+r)^{2n-1}\left((1+r)+\frac{\lambda_n}{2(1+r)^{n-1}}\right)\mathbb{E}\left(1-\sum_{i=0}^{n-1}\tau_i|\mathcal{H}_0\right)\right) \\
&\quad -\frac{\lambda_n}{2}\mathbb{E}\left(\mathbb{E}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^Q)|\mathcal{H}_0\right)\right)-(1+c)^2 \\
&= (1+r)^{2n}\,\mathbb{E}\left(1-\sum_{i=0}^{n-1}\tau_i\right) \\
&\quad +\frac{\lambda_n}{2}\left((1+r)^n-(1+c)-(1+r)^n\,\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)\right)-(1+c)^2.
\end{aligned}
$$

Then substituting the value of λ_n we have

$$
\begin{aligned}
&\mathbb{COV}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P),\Pi_{i=1}^n(1+\widetilde{R}_i^Q)\right) \\
&= (1+r)^{2n}\,\mathbb{E}\left(1-\sum_{i=0}^{n-1}\tau_i\right) \\
&\quad +\left(\frac{(1+r)^n-(1+c)}{\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)}-(1+r)^n\right) \\
&\qquad\qquad \times\left((1+r)^n-(1+c)-(1+r)^n\,\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)\right) \\
&\quad -(1+c)^2 \\
&= ((1+r)^n-(1+c))^2\left[\frac{1}{\mathbb{E}\left(\sum_{i=0}^{n-1}\tau_i\right)}-1\right] \\
&= \mathbb{VAR}\left(\Pi_{i=1}^n(1+\widetilde{R}_i^P)\right).
\end{aligned}
$$

Therefore the strategy $\{\omega_k\}$ gives us a global extremum.

4.7 Proof of corollary 2.10

First from remark 2.7, we deduce that

$$
\begin{aligned}
&\frac{\mathbb{COV}\left(\Pi_{i=k}^n(1+\widetilde{R}_i^P),\Pi_{i=k}^n(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right)}{\mathbb{VAR}\left(\Pi_{i=k}^n(1+\widetilde{R}_i^P)|\mathcal{H}_{k-1}\right)} \\
&= \frac{(1+r)^{n-k+1}-\mathbb{E}\left(\Pi_{i=k}^n(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right)}{\left((1+r)^{n-k+1}+\frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right]}.
\end{aligned}
$$

Thus there comes

$$
\mathbb{E}\left(\Pi_{i=k}^n(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right)-(1+r)^{n-k+1}
$$

$$
= -\frac{\mathbb{COV}\left(\Pi_{i=k}^{n}(1+\widetilde{R}_i^P), \Pi_{i=k}^{n}(1+\widetilde{R}_i^Q)|\mathcal{H}_{k-1}\right)}{\mathbb{VAR}\left(\Pi_{i=k}^{n}(1+\widetilde{R}_i^P)|\mathcal{H}_{k-1}\right)}
\times\left((1+r)^{n-k+1}+\frac{\lambda_n}{2\Pi_{i=1}^{k-1}(1+\widetilde{R}_i^P)}\right)\mathbb{E}\left[\sum_{i=k-1}^{n-1}\tau_i|\mathcal{H}_{k-1}\right].
$$

Using proposition 2.8 with $k=1$, we finally have

$$
\begin{aligned}
&\mathbb{E}\left(\Pi_{i=1}^{n}(1+\widetilde{R}_i^Q)\right)-(1+r)^n \\
= \quad & -\beta_n\left((1+r)^n+\frac{\lambda_n}{2}\right)\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right] \\
= \quad & -\beta_n\left((1+r)^n+\left(\frac{(1+r)^n-(1+c)}{\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right]}-(1+r)^n\right)\right)\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right] \\
= \quad & \beta_n\left(\frac{(1+c)-(1+r)^n}{\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right]}\right)\mathbb{E}\left[\sum_{i=0}^{n-1}\tau_i\right] \\
= \quad & \beta_n\left((1+c)-(1+r)^n\right) \\
= \quad & \beta_n\left(\mathbb{E}\left(\Pi_{i=1}^{n}(1+\widetilde{R}_i^P)\right)-(1+r)^n\right).
\end{aligned}
$$

5 CONCLUSION

We have presented a multi-period mean variance analysis in portfolio selection in the case where the investor's portfolio consists of a single stock and bond and where only fairly general conditions are imposed on these assets.

The closed-form solution to the multi-period mean-variance problem was constructed using optimization techniques in an abstract L^2-space setting as well as conditioning properties of random variables and backward induction. The advantage of the proposed solution is that it is general enough to allow for the incorporation of time dependence in modelling the rate of return, as well as dependence, if one so wishes, on exogenous variables, such as economic factors that might have the property to improve substantially our ability to predict future rate of return. As a corollary, a security market line result in a multiperiod capital asset pricing model was derived.

We have also exhibited several rich classes of models for the excess rate of return of a risky asset. These models were selected in order to satisfy certain basic criteria, namely simplicity of use, computational efficiency and interpretability. Many of them offer a simplified symbolic representation for the optimal values of the parameters defining the solution to the multiperiod mean-variance portfolio and have be seen to afford computational implementations with tractability throughout the calculation process, thereby ensuring accurate results in real time, even when parameter estimation is required.

Upcoming investigations will include the extension of the multi-period mean-variance portfolio solution to the case where the investor's portfolio consists of many risky assets and the interest rate varies in time. Also constrained multi-period mean-variance should be considered, for example by prohibiting short sales.

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