subfBm (15/06/04)

Sub-fractional Brownian motion and its relation to occupation times

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Abstract

We study a long-range dependence Gaussian process which we call "sub-fractional Brownian motion" (sub-fBm), because it is intermediate between Brownian motion (Bm) and fractional Brownian motion (fBm) in the sense that it has properties analogous to those of fBm, but the increments on non-overlapping intervals are more weakly correlated and their covariance decays polynomially at a higher rate. Sub-fBm has a parameter $h \in (0,2)$, we show how it arises from occupation time fluctuations of branching particle systems for $h \geq 1$ and we exhibit the long memory effect of the initial condition.

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1. Introduction

Fractional Brownian motion (fBm) $\xi^h = \{\xi^h(t), t \geq 0\}$ is the best known and most widely used (centered) Gaussian process with long-range dependence. Its covariance function is

$$R_h(s,t) = \frac{1}{2}(s^h + t^h - |s-t|^h), \tag{1.1}$$

where $h \in (0,2)$, and the case $h = 1$ corresponds to Brownian motion (Bm). $H = h/2$ is called Hurst parameter. fBm is the only Gaussian process which is self-similar and has stationary increments. For basic information on fBm see Samorodnitsky and Taqqu (1994).

In this paper we study a (centered) Gaussian process $\zeta^h = \{\zeta^h(t), t \geq 0\}$ with covariance function

$$C_h(s,t) = s^h + t^h - \frac{1}{2}[(s+t)^h + |s-t|^h], \tag{1.2}$$

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where $h \in (0,2)$. The case $h = 1$ also corresponds to Brownian motion. For $h \neq 1$ this process has some of the main properties of fBm, but its increments are not stationary (see however Remark (b) on the Theorem in Section 2), they are more weakly correlated on non-overlapping intervals, and their covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. Hence ζ^h may be considered as being intermediate between Bm and fBm, and for this reason we call it "sub-fractional Brownian motion" (sub-fBm). Our main objective in this paper is to study the properties of sub-fBm and compare them to those of fBm. As long-range dependence is an important feature of stochastic models in several areas (e.g., hydrology, turbulence, telecommunications, financial markets), we hope that the intermediate properties of sub-fBm may make it useful in some applications.

We first found sub-fBm with $h \geq 1$ in connection with occupation time fluctuations of a branching particle system with Poisson initial condition, where the particle motion is a symmetric α-stable Lévy process on $\mathbb{R}^d, \alpha \in (0,2]$ $\quad (\alpha = 2$ corresponds to Brownian motion). Similarly, for $h \geq 1$ fBm arises from occupation time fluctuations of a particle system without branching. Special cases of these results (with $h = 3/2$) have been obtained by Deuschel and Wang (1994) for Brownian motion without branching, and by Iscoe (1986) with branching in the context of super-Brownian motion, but no mention is made in those papers on long-range dependence processes. For our objectives in this paper it suffices to prove convergence of the covariances of the occupation time fluctuation processes. From the perspective of the behavior of the particle systems it is necessary to prove stronger results, namely, functional convergence of the occupation time fluctuation processes. This requires different and more specialized tools, and it is carried out in a separate paper (Bojdecki et al, 2004). The relevant point in this paper is the fact that the temporal structures of the limit covariances are of types (1.1) or (1.2).

In Section 2 we study the properties of sub-fBm. In Section 3 we show how, for $h \geq 1$, fBm and sub-fBm arise from occupation time fluctuations of particle systems, and we stress the long-range effect of the initial condition of the branching system.

2. Existence and properties of sub-fractional Brownian motion

Sub-fBm with $h = 1$ corresponds to Brownian motion, and the cases $h = 0$ and $h = 2$ are irrelevant (however, for $h = 2$ the covariance (1.2) belongs to the null process, whereas in this case fBm is of the form $\xi(t) = t\xi(1)$). Hence we restrict h to the interval $(0,2)$.

Existence of sub-fBm ζ^h for any $h \in (0,2)$ can be shown in the following two ways: (1) Consider the process

$$\frac{1}{\sqrt{2}}(\xi^h(t) + \xi^h(-t)), \quad t \geq 0, \tag{2.1}$$

where $\{\xi^h(t), -\infty < t < \infty\}$ is fBm on the whole real line. It is easy to see using (1.1) that the covariance of the process (2.1) is precisely $C_h(s,t)$ given by (1.2). Therefore ζ^h exists. (2) It can be shown by means of the fractional Laplacian $\Delta_h = -(-\Delta)^{h/2}$ that the function $C_h(s,t)$ defined by (1.2) is non-negative definite. Therefore a Gaussian random field on $\mathbb{R}$ with $C_h(s,t)$ as its covariance exists. The proof is analogous to the one in Bojdecki and Gorostiza (1999). Existence of ζ^h for $h \geq 1$ also follows from the results in Section 3, where $C_h(s,t)$ appears as a limit of covariances, and therefore is non-negative definite.

In the following theorem and subsequent remarks we give the properties of sub-fBm ζ^h and comparisons with fBm. Recall that ξ^h denotes fBm and $R_h(s,t)$ its covariance.

Theorem. *Sub-fBm ζ^h, $h \in (0,2)$, has the following properties:*

(1) Self-similarity:

$$\{\zeta^h(at), t \geq 0\} \stackrel{d}{=} \{a^{h/2}\zeta^h(t), t \geq 0\} \quad for\ each \quad a > 0.$$

(2) Covariance: For all $s,t > 0$,

$$\begin{aligned} C_h(s,t) &> R_h(s,t) \quad \text{if} \quad h < 1, \\ C_h(s,t) &< R_h(s,t) \quad \text{if} \quad h > 1, \\ C_h(s,t) &> 0. \end{aligned} \tag{2.2}$$

(3) Second moment of increments: For all $s \leq t$,

$$E(\zeta^h(t) - \zeta^h(s))^2 = -2^{h-1}(t^h + s^h) + (t+s)^h + (t-s)^h, \tag{2.3}$$

$$(2 - 2^{h-1})(t-s)^h \leq E(\zeta^h(t) - \zeta^h(s))^2 \leq (t-s)^h \quad if \quad h > 1, \tag{2.4}$$

$$(t-s)^h \leq E(\zeta^h(t) - \zeta^h(s))^2 \leq (2 - 2^{h-1})(t-s)^h \quad if \quad h < 1, \tag{2.5}$$

and the constants in these inequalities are the best possible.

(4) Hölder continuity: ζ^h has a continuous version for each h, and for each $0 < \varepsilon < h/2$ and each $T > 0$ there exists a random variable $K_{\varepsilon,T}$ such that

$$|\zeta^h(t) - \zeta^h(s)| \leq K_{\varepsilon,T}|t-s|^{h/2-\varepsilon}, \quad s,t \in [0,T], \quad a.s. \tag{2.6}$$

(5) Correlation of increments: For $0 \leq u < v \leq s < t$, let

$$R_{u,v,s,t} = E(\xi^h(v) - \xi^h(u))(\xi^h(t) - \xi^h(s)), \tag{2.7}$$

and

$$C_{u,v,s,t} = E(\zeta^h(v) - \zeta^h(u))(\zeta^h(t) - \zeta^h(s)), \tag{2.8}$$

then

$$\begin{aligned} C_{u,v,s,t} &= \frac{1}{2}[(t+u)^h + (t-u)^h + (s+v)^h + (s-v)^h \\ &\qquad -(t+v)^h - (t-v)^h - (s+u)^h - (s-u)^h], \end{aligned} \tag{2.9}$$

$$C_{u,v,s,t} > 0 \quad if \quad h > 1. \tag{2.10}$$

and

$$C_{u,v,s,t} < 0 \quad if \quad h < 1. \tag{2.11}$$

If $D_{u,v,s,t}$ is defined by

$$C_{u,v,s,t} = R_{u,v,s,t} + D_{u,v,s,t}, \tag{2.12}$$

then

$$D_{u,v,s,t} < 0 \quad if \quad h > 1, \tag{2.13}$$

and

$$D_{u,v,s,t} > 0 \quad if \quad h < 1. \tag{2.14}$$

For $u \geq 0, r > 0$ let $\rho^{\xi^h}_{u,r}$ and $\rho^{\zeta^h}_{u,r}$ denote the correlation coefficients of $\xi^h_{u+r} - \xi^h_u, \xi^h_{u+2r} - \xi^h_{u+r}$ and $\zeta^h_{u+r} - \zeta^h_u, \zeta^h_{u+2r} - \zeta^h_{u+r}$, respectively. Then

$$|\rho^{\zeta^h}_{u,r}| \leq |\rho^{\xi^h}_{u,r}|. \tag{2.15}$$

$$\lim_{s,t\to\infty} C_{u,v,s,t} = 0, \tag{2.16}$$

$$C_{u,v,s,t} < 2^{h-2}v^h \quad if \quad h > 1, \tag{2.17}$$

$$C_{u,v,s,t} > -\frac{1}{2}v^h \quad if \quad h < 1, \tag{2.18}$$

$$C_{u,v,s+\tau,t+\tau} \sim \frac{h(h-1)(2-h)}{2}(t-s)(v^2-u^2)\tau^{h-3} \ as\ \tau\to\infty\ if\ h\neq 1. \tag{2.19}$$

(6) ζ^h is not a Markov process if $h \neq 1$.

(7) ζ^h is not a semimartingale if $h \neq 1$.

(8) Integral representations (for $h \neq 1$):
Moving average representation:

$$\zeta^h(t) = \frac{1}{C_1(h)}\int_{\mathbb{R}}[((t-s)^+)^{(h-1)/2} + ((t+s)^-)^{(h-1)/2} - 2((-s)^+)^{(h-1)/2}]dW(s),$$

where W is the Brownian measure on $\mathbb{R}$ and

$$C_1(h) = \left[2\left(\int_0^\infty ((1+s)^{(h-1)/2} - s^{(h-1)/2})^2 ds + \frac{1}{h}\right)\right]^{1/2}.$$

Spectral representation:

$$\zeta^h(t) = \frac{1}{C_2(h)}\int_{\mathbb{R}} \frac{\cos(ts)-1}{is}|s|^{(1-h)/2}d\widetilde{W}(s),$$

where $\widetilde{W} = W^{(1)} + iW^{(2)}$ is a complex Gaussian measure on $\mathbb{R}$ such that

$$W^{(1)}(A) = W^{(1)}(-A),\ \ W^{(2)}(A) = -W^{(2)}(-A),\ \ E(W^{(1)}(A))^2 = E(W^{(2)}(A))^2 = \frac{1}{2}|A|,$$

(here $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$), and

$$C_2(h) = \left(\frac{\pi}{h\Gamma(h)\sin(\pi h/2)}\right)^{1/2}.$$

Remarks

(a) The increments of fBm are self-similar in the sense that

$$\xi^h(t+a\tau) - \xi^h(t) \stackrel{d}{=} a^{h/2}(\xi^h(t+\tau) - \xi^h(t)) \quad \text{for each} \quad a > 0,$$

but sub-fBm does not have this property.

(b) Clearly sub-fBm does not have stationary increments, but this property is replaced by the inequalities (2.4) and (2.5).

(c) Recall that $R_{u,v,s,t} > 0$ (resp. < 0) if $h > 1$ (resp. < 1). Formulas (2.7), (2.8), (2.10), (2.11), (2.12), (2.13) and (2.14) show that the covariances of increments of sub-fBm on non-overlapping intervals have the same sign but are smaller in absolute value than those of fBm. (2.15) shows that the increments of sub-fBm on intervals $[u, u+r], [u+r, u+2r]$ are more weakly correlated than those of fBm. Moreover, for $h < 1$ it can be shown, with essentially the same proof, that the same property holds for arbitrary intervals $[u, v], [s, t]$, with $v \leq s$, and it is also true (with a longer proof) for $h > 1$ and intervals $[0, v], [s, t]$. We have not been able to prove it for $h > 1$ and arbitrary $[u, v], [s, t], v \leq s$.

(d) In contrast to (2.16), for fBm $\lim_{s,t\to\infty} R_{u,v,s,t} = 0$ only for $0 < h \leq 1$.

(e) Inequalities (2.17) and (2.18) show that the upper end point of the lower interval (u, v) has a dominating effect on the correlation $C_{u,v,s,t}$ if s is not large. By (2.16) this effect becomes irrelevant for large s.

(f) Inequality (2.17) implies that for $h > 1$,

$$\limsup_{v\to 0} \frac{1}{v^h} C_{0,v,s,t} \leq 1,$$

whereas for fBm, from (2.25) below, we have

$$\lim_{v\to 0} \frac{1}{v^h} R_{0,v,s,t} = \infty.$$

This means that the increments on the intervals $[0, v]$ and $[s, t]$ approach independence as $v \to 0$ faster for sub-fBm than for fBm, and in the case of sub-fBm this occurs uniformly with respect to the intervals $[s, t]$, $s > \delta > 0$.

(g) fBm and sub-fBm become similar for large t in the sense that for each $\tau > 0$,

$$\lim_{t\to\infty} \frac{C_h(t, t+\tau)}{R_h(t, t+\tau)} = 2 - 2^{h-1},$$

but on the other hand

$$\lim_{t\to\infty} t^{-h}[C_h(t, t+\tau) - R_h(t, t+\tau)] = 1 - 2^{h-1}.$$

(h) In contrast to (2.19), $R_{u,v,s+\tau,t+\tau} \sim \frac{h(h-1)}{2}(t-s)(v-u)\tau^{h-2}$ as $\tau \to \infty$ if $h \neq 1$. Thus the long-range dependence decays at a higher rate for sub-fBm than for fBm. This, together with Remark (c), justifies the name sub-fractional Brownian motion for ζ^h.

(i) The method of proof for (7) can also be used to show that fBm is not a semimartingale if $h \neq 1$ (see Bojdecki et al, 2002, Lemma 2.5).

Proof of the Theorem

(1) The self-similarity is obvious from the form of $C_h(s, t)$ in (1.2).

(2) The first two assertions are obvious. It follows from (2.3) (which is an immediate consequence of (1.2)) that

$$2^{h-1}(t^h + s^h) \leq (t+s)^h + (t-s)^h, \quad s < t$$

Applying this inequality with $t' = t+s$ and $s' = t-s$ we have

$$(t+s)^h + (t-s)^h < 2(t^h + s^h), \quad 0 < s < t,$$

which proves (2.2) by (1.2).

(3) (2.3) Let $h > 1$. Then
$$(t+s)^h \le 2^{h-1}(t^h + s^h).$$
Hence $2^{h-1}(t^h + s^h) - (t+s)^h \ge 0$, and we look for the maximum $\delta \ge 0$ and the minimum $\gamma > 0$ such that
$$\gamma(t-s)^h \ge 2^{h-1}(t^h + s^h) - (t+s)^h \ge \delta(t-s)^h. \tag{2.20}$$
Denote $x = t - s \ge 0$ and consider
$$\gamma x^h \ge 2^{h-1}((s+x)^h + s^h) - (2s+x)^h \ge \delta x^h.$$
The function $x \mapsto 2^{h-1}((s+x)^h + s^h) - (2s+x)^h - \gamma x^h$ is 0 at $x = 0$, so we look for the minimum γ such that this function decreases for $x > 0$, and we find
$$\min \gamma = 2^{h-1} - 1. \tag{2.21}$$
By a similar argument we find
$$\max \delta = 0. \tag{2.22}$$
Therefore (2.21) and (2.22) are the best constants for (2.20), and (2.4) follows from (2.3) and (2.20).

Now let $h < 1$. Then
$$(t+s)^h \ge 2^{h-1}(t^h + s^h),$$
and we look for the maximum δ and the minimum γ such that
$$\gamma(t-s)^h \ge (t+s)^h - 2^{h-1}(t^h + s^h) \ge \delta(t-s)^h.$$
Proceeding similarly as above we find (2.5).

(4) (2.6) is a consequence of (2.4) and (2.5) by Kolmogorov's criterion, but of course this also follows from the representation (2.1) of ζ^h.

(5) We have from (2.8)
$$C_{u,v,s,t} = C_h(v,t) - C_h(v,s) - C_h(u,t) + C_h(u,s),$$
so (2.9) follows from (1.2). Write (2.9) as
$$C_{u,v,s,t} = \frac{1}{2}(g(t) - g(s)) \tag{2.23}$$
where
$$g(t) = -(t+v)^h - (t-v)^h + (t+u)^h + (t-u)^h. \tag{2.24}$$
For $h \in (1,2)$ the function $x \mapsto x^{h-1}$ is concave, hence
$$(t+v)^{h-1} + (t-v)^{h-1} < (t+u)^{h-1} + (t-u)^{h-1},$$
which implies that $\frac{d}{dt}g(t) > 0$. Thus $C_{u,v,s,t}$ increases for $t > s$, and since it vanishes at $t = s$, then (2.10) holds.

An analogous argument proves (2.11), using this time the fact that the function $x \mapsto x^{h-1}$ is convex for $h \in (0,1)$.

The covariance (2.7) of the increments of fBm is given by

$$R_{u,v,s,t} = \frac{1}{2}[(t-u)^h - (t-v)^h + (s-v)^h - (s-u)^h], \tag{2.25}$$

hence from (2.9) and (2.12) we have

$$D_{u,v,s,t} = \frac{1}{2}[(t+u)^h - (t+v)^h + (s+v)^h - (s+u)^h].$$

Writing

$$D_{u,v,s,t} = \frac{1}{2}(f(t) - f(s)), \quad t \geq s,$$

where $f(t) = (t+u)^h - (t+v)^h$, we see that the function $t \mapsto D_{u,v,s,t}$ decreases if $h > 1$ and increases if $h < 1$, and since it vanishes at $t = s$, then (2.13) and (2.14) follow.

We have $\rho^{\xi^h}_{u,r} = r^{-h} R_{u,u+r,u+r,u+2r}$ since $E(\xi^h(t) - \xi^h(s))^2 = |t-s|^h$, and

$$\rho^{\zeta^h}_{u,r} = \frac{C_{u,u+r,u+r,u+2r}}{(V_{u,u+r} V_{u+r,u+2r})^{1/2}},$$

where $V_{s,t} = E(\zeta^h(t) - \zeta^h(s))^2$. By (2.10)-(2.14) and the fact that $R > 0$ if $h > 1$ and $R < 0$ if $h < 1$, it is easily seen that (2.15) is equivalent to

$$1 + \frac{D_{u,u+r,u+r,u+2r}}{R_{u,u+r,u+r,u+2r}} \leq \frac{(V_{u,u+r} V_{u+r,u+2r})^{1/2}}{r^h}. \tag{2.26}$$

If $h < 1$ then the left hand side is smaller than 1 (by (2.14)) and the right hand side is bigger than 1 (by (2.5)), hence (2.26) follows.

Now let $h > 1$. By (2.3), (2.9), (2.25), putting $x = 2u/r$ we obtain that (2.26) is equivalent to

$$\begin{aligned} &2^h - 2 + 2(x+2)^h - (x+1)^h - (x+3)^h \\ &\leq (2^{h-1} - 1)[2 + 2(x+1)^h - x^h - (x+2)^h]^{1/2}[2 + 2(x+3)^h - (x+2)^h - (x+4)^h]^{1/2}. \end{aligned}$$

Denote $a(x) = (x+2)^h + x^h - 2(x+1)^h, x \geq 0$. We have to prove that

$$a(0) - a(x+1) \leq \frac{1}{2} a(0)[2 - a(x)]^{1/2}[2 - a(x+2)]^{1/2}$$

for all $x \geq 0$. Since a is decreasing it suffices to prove that

$$a(0) - a(x+1) \leq \frac{1}{2} a(0)(2 - a(x)),$$

or

$$\frac{a(x)}{a(x+1)} \leq \frac{2}{a(0)}. \tag{2.27}$$

Observe that the function $x \mapsto a(x)/a(x+1)$ is decreasing on $\mathbb{R}_+$. Indeed, it is easily seen that $(-1)^n \frac{d^n}{dx^n} a(x) \geq 0$ for $n = 0, 1, 2, \ldots$ (recall that $1 < h < 2$), hence, by Bernstein's theorem a is the Laplace transform of some positive measure on $\mathbb{R}_+$ (see e.g. Feller, 1971). Then, by the Schwarz inequality $a(\frac{x+y}{2}) \leq [a(x)a(y)]^{1/2}$, and this inequality implies that $\log a(x)$ is a convex function. So $\frac{d}{dx} \log a(x) \leq \frac{d}{dx} \log a(x+1)$, which yields $\frac{d}{dx} \frac{a(x)}{a(x+1)} \leq 0$.

As a consequence we have $a(x)/a(x+1) \leq a(0)/a(1)$ for $x \geq 0$, so to obtain (2.27) it suffices to prove that $a(0)/a(1) \leq 2/a(0)$ or, explicitly, $(1/2)4^h \leq 3^h - 1$, which can be verified by calculus (the function $h \mapsto 3^h - 1 - (1/2)4^h$ is concave on $[1,2]$, equal to zero at 1 and 2). So (2.15) is proved.

(2.16) can be proved from (2.9) by Taylor's formula or by l'Hôpital's rule.

Let $h > 1$. Using the notation (2.23) and (2.24) we have

$$C_{u,v,s,t} - 2^{h-2}v^h = \frac{1}{2}(g(t) - g(s) - 2^{h-1}v^h).$$

Since $g(t)$ is increasing, in order to prove (2.17) it is enough to show that

$$\lim_{t\to\infty}(g(t) - g(s)) \leq 2^{h-1}v^h,$$

but $\lim_{t\to\infty} g(t) = 0$ (proof of (2.16)), so it remains to show that

$$-g(s) \leq 2^{h-1}v^h. \tag{2.28}$$

The function $x \mapsto (1+x)^h + (1-x)^h - 2 - 2^{h-1}x^h, 0 \leq x < 1$, is decreasing, hence

$$(s+v)^h + (s-v)^h \leq 2s^h + 2^{h-1}v^h.$$

and by convexity

$$-(s+v)^h - (s-v)^h \leq -2s^h.$$

Hence (2.28) is true and (2.17) is proved.

(2.18) is proved similarly.

(2.19) can be proved by l'Hôpital's rule.

(6) ζ^h is not Markovian because the covariance $C_h(s,t)$ in (1.2) does not have triangular property (e.g. Neveu, 1968).

(7) The non-semimartingale property of ζ^h follows from the inequalities (2.4) and (2.5), which imply that (on a bounded time interval) for $h > 1$ the quadratic variation of ζ^h is 0 and the trajectories have infinite variation, and for $h < 1$ the quadratic variation of ζ^h is infinite. This is a consequence of a general result proved in Bojdecki et al (2002), Lemma 2.5.

(8) The integral representations for ζ^h follow from the corresponding ones for fBm ξ^h (Samorodnitsky and Taqqu, 1994; see also Nualart, 2003) and (2.1).

3. Occupation time fluctuations

In this section we show how fBm ξ^h and sub-fBm ζ^h arise from occupation time fluctuations of some particle systems for $h \geq 1$.

Consider a particle system in $\mathbb{R}^d$ where the particles independently migrate according to a symmetric α-stable Lévy process ($\alpha \in (0,2]$) and branch at rate V according to a critical binary branching law, and such that at the initial time 0 the particles are distributed by a Poisson random measure with intensity λ (Lebesgue measure). We denote by N_t the empirical measure of the system at time t, i.e., N_t is a random point measure on $\mathbb{R}^d$ such that $N_t(A)$ is the number of particles present in the set A at time t. We write $\langle \mu, f \rangle = \int f d\mu$ where μ is a measure and

f a measurable function. We consider test functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (the usual space of rapidly decreasing C^∞ functions).

For $T > 0$, let $L_T(t)$ denote the occupation time of the system at time Tt, i.e.

$$\langle L_T(t), \varphi \rangle = \int_0^{Tt} \langle N_s, \varphi \rangle ds = T \int_0^t \langle N_{Ts}, \varphi \rangle ds, \quad t \geq 0.$$

The parameter T will tend to ∞ and t is the time variable of the process. Due to criticality of the branching and invariance of λ for the semigroup of the motion we have

$$E\langle L_T(t), \varphi \rangle = Tt\langle \lambda, \varphi \rangle.$$

We define the occupation time fluctuation process $X_T = \{X_T(t), t \geq 0\}$ by

$$\begin{aligned} \langle X_T(t), \varphi \rangle &= \frac{1}{F_T}(\langle L_T(t), \varphi \rangle - Tt\langle \lambda, \varphi \rangle) \\ &= \frac{T}{F_T} \int_0^t (\langle N_{Ts}, \varphi \rangle - \langle \lambda, \varphi \rangle) ds, \quad t \geq 0, \end{aligned} \tag{3.1}$$

where F_T is a norming. The task concerning the behavior of the particle system consists in proving that for an appropriate F_T, the process X_T converges in distribution as $T \to \infty$ to a Gaussian process in the function space $C([0, \tau], \mathcal{S}'(\mathbb{R}^d))$ for any $\tau > 0$, where $\mathcal{S}'(\mathbb{R}^d)$ is the space of tempered distributions. This is done in Bojdecki et al (2004) for the cases of long-range dependence. However, for our purpose in this paper it suffices to compute the covariance of X_T (which tells what F_T should be) and obtain its limit as $T \to \infty$.

Taking covariance in (3.1) we have

$$\mathrm{Cov}(\langle X_T(s), \varphi \rangle, \langle X_T(t), \psi \rangle) = \frac{T^2}{F_T^2} \int_0^s du \int_0^t dv \; \mathrm{Cov}(\langle N_{Tu}, \varphi \rangle, \langle N_{Tv}, \psi \rangle). \tag{3.2}$$

The covariance of the empirical measure process N is given by

$$\begin{aligned} \mathrm{Cov}(\langle N_u, \varphi \rangle, \langle N_v, \psi \rangle) &= \langle \lambda, \varphi \mathcal{T}_{v-u} \psi \rangle \\ &+ V \int_0^u \langle \lambda, \varphi \mathcal{T}_{u+v-2r} \psi \rangle dr, \quad u \leq v, \end{aligned} \tag{3.3}$$

where $\mathcal{T}_t$ is the semigroup of the α-stable process (derivations of covariance formulas of this type can be found e.g. in Gorostiza, 1983, and Gorostiza and Rodrigues, 1999). The model without the branching corresponds to $V = 0$.

We now express the covariance (3.3) by means of the Fourier transform. Recall that the Fourier transform $\widehat{\varphi}$ of $\varphi \in \mathcal{S}(\mathbb{R}^d)$ is defined by

$$\widehat{\varphi}(z) = \int_{\mathbb{R}^d} e^{ix \cdot z} \varphi(x) dx,$$

($x \cdot z$ is the inner product in $\mathbb{R}^d$). Using the formulas

$$\langle \lambda, \varphi \psi \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\varphi}(z) \overline{\widehat{\psi}(z)} dz,$$

and

$$\widehat{\mathcal{T}_t \varphi}(z) = e^{-t|z|^\alpha} \widehat{\varphi}(z),$$

we obtain from (3.3)

$$
\begin{aligned}
&\mathrm{Cov}(\langle N_u,\varphi\rangle,\langle N_v,\psi\rangle)\\
&= \frac{1}{(2\pi)^d}\left(\int_{\mathbb{R}^d}\widehat{\varphi}(z)\overline{(\mathcal{T}_{v-u}\psi)\widehat{\ }(z)}dz+V\int_0^u\int_{\mathbb{R}^d}\widehat{\varphi}(z)\overline{(\mathcal{T}_{u+v-2r}\psi)\widehat{\ }(z)}dzdr\right)\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}\left(e^{-(v-u)|z|^\alpha}+V\int_0^u e^{-(u+v-2r)|z|^\alpha}dr\right)dz\\
&= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}\left(e^{-(v-u)|z|^\alpha}+V\frac{e^{-(v-u)|z|^\alpha}-e^{-(v+u)|z|^\alpha}}{2|z|^\alpha}\right)dz,\ u\le v, \qquad (3.4)
\end{aligned}
$$

and substituting (3.4) into (3.2) we have

$$
\begin{aligned}
&\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\\
&= \frac{1}{(2\pi)^d}\frac{T^2}{F_T^2}\left(\int_0^s du\int_s^t dv+2\int_0^s dv\int_0^v du\right)\int_{\mathbb{R}^d}\hat{\varphi}(z)\overline{\hat{\psi}(z)}\\
&\quad\cdot\left(e^{-T(v-u)|z|^\alpha}+V\frac{e^{-T(v-u)|z|^\alpha}-e^{-T(v+u)|z|^\alpha}}{2|z|^\alpha}\right)dz,\ s\le t. \qquad (3.5)
\end{aligned}
$$

We have to take the limit in (3.5) as $T\to\infty$ for the systems without branching ($V=0$) and with branching ($V>0$) and with appropriate dimensions d.

Case $V=0$ (no branching).

Let $d<\alpha$.

Making the change of variable $z=z_T(y)=T^{-1/\alpha}(v-u)^{-1/\alpha}y$ in (3.5) we have

$$
\begin{aligned}
&\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\\
&= \frac{1}{(2\pi)^d}\frac{T^{2-d/\alpha}}{F_T^2}\left(\int_0^s du\int_s^t dv+2\int_0^s dv\int_0^v du\right)(v-u)^{-d/\alpha}\\
&\quad\cdot\int_{\mathbb{R}^d}\widehat{\varphi}(z_T(y))\overline{\widehat{\psi}(z_T(y))}e^{-|y|^\alpha}dy. \qquad (3.6)
\end{aligned}
$$

To obtain convergence in (3.6) we put

$$F_T=T^{1-d/2\alpha}, \qquad (3.7)$$

and then

$$
\begin{aligned}
&\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\\
&\to \frac{1}{(2\pi)^d}\widehat{\varphi}(0)\overline{\widehat{\psi}(0)}\int_{\mathbb{R}^d}e^{-|y|^\alpha}dy\left(\int_0^s du\int_s^t dv+2\int_0^s dv\int_0^v du\right)(v-u)^{-d/\alpha}\\
&= \langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{\Gamma(d/\alpha)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)(1-d/\alpha)(2-d/\alpha)}(t^{2-d/\alpha}+s^{2-d/\alpha}-(t-s)^{2-d/\alpha})
\end{aligned}
$$

as $T\to\infty$. So,

$$
\begin{aligned}
&\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\\
&\to \langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}(t^h+s^h-(t-s)^h),\quad s\le t, \qquad (3.8)
\end{aligned}
$$

as $T \to \infty$, with $h = 2 - d/\alpha$.

The possible values of h are $h \in (1, 3/2]$ for $\alpha \in (1, 2]$ and $d = 1$. The value $h = 3/2$ corresponds to Brownian particle motion ($\alpha = 2$).

The temporal part of (3.8) has the form of the covariance $R_h(s,t)$ of fBm in (1.1) with $h \in (1, 3/2]$.

Now let $d > \alpha$. This case does not lead to long-range dependence, but it is needed in the proof for the branching system.

Doing the integration on v in (3.5) we have

$$\begin{aligned} &\mathrm{Cov}(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\ &= \frac{1}{(2\pi)^d} \frac{T}{F_T^2} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{|z|^\alpha} \int_0^s (2 - e^{-Tu|z|^\alpha}(1 + e^{-T(t-s+u)|z|^\alpha}))du dz. \end{aligned} \tag{3.9}$$

To obtain convergence in (3.9) we put

$$F_T = T^{1/2} \tag{3.10}$$

and then we have

$$\begin{aligned} &\mathrm{Cov}(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\ &\to \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\widehat{\varphi}(z)\overline{\widehat{\psi}(z)}}{|z|^\alpha} dz 2s = 2\langle \lambda, \varphi G\psi\rangle s, \quad s \le t, \end{aligned} \tag{3.11}$$

as $T \to \infty$, where G is the potential operator of the symmetric α-stable process,

$$G\psi(x) = \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{d/2}\Gamma(\frac{\alpha}{2})} \int_{\mathbb{R}^d} \frac{\psi(y)}{|x-y|^{d-\alpha}} dy, \ d > \alpha. \tag{3.12}$$

The temporal part of the limit (3.11) is Bm.

Case $V > 0$ (branching).

Let $\alpha < d < 2\alpha$.

In (3.5), using (3.9) for the first term and making the changes of variable $z = z_T(y) = T^{-1/\alpha}(v-u)^{-1/\alpha}y$ in the second term and $z = w_T(y) = T^{-1/\alpha}(v+u)^{-1/\alpha}y$ in the third term, we have

$$\begin{aligned} &\mathrm{Cov}(\langle X_T(s), \varphi\rangle, \langle X_T(t), \psi\rangle) \\ &= \frac{1}{(2\pi)^d} \frac{T}{F_T^2} \int_{\mathbb{R}^d} \frac{\hat{\varphi}(z)\overline{\hat{\psi}(z)}}{|z|^\alpha} \int_0^s (2 - e^{-Tu|z|^\alpha}(1 + e^{-T(t-s+u)|z|^\alpha}))du dz \\ &+ \frac{1}{(2\pi)^d} \left(\int_0^s du \int_s^t dv + 2\int_0^s dv \int_0^v du \right) \\ &\quad \frac{V}{2} \frac{T^{3-d/\alpha}}{F_T^2} \left((v-u)^{1-d/\alpha} \int_{\mathbb{R}^d} \hat{\varphi}(z_T(y))\overline{\hat{\psi}(z_T(y))} \frac{e^{-|y|^\alpha}}{|y|^\alpha} dy \right. \\ &\quad \left. -(v+u)^{1-d/\alpha} \int_{\mathbb{R}^d} \hat{\varphi}(w_T(y))\overline{\hat{\psi}(w_T(y))} \frac{e^{-|y|^\alpha}}{|y|^\alpha} dy \right). \end{aligned} \tag{3.13}$$

To obtain convergence in (3.13) we put

$$F_T = T^{(3-d/\alpha)/2} \tag{3.14}$$

and then, since the limit (3.11) holds with F_T given by (3.10), we have

$$\begin{aligned}
&\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\\
\to\;& \frac{V}{2}\frac{1}{(2\pi)^d}\widehat{\varphi}(0)\overline{\widehat{\psi}(0)}\int_{\mathbb{R}^d}\frac{e^{-|y|^\alpha}}{|y|^\alpha}dy\\
&\cdot\left(\int_0^s du\int_s^t dv+2\int_0^s dv\int_0^v du\right)((v-u)^{1-d/\alpha}-(v+u)^{1-d/\alpha})\\
=\;& \langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{V\Gamma(2-h)}{2^{d-1}\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}\left(t^h+s^h-\frac{1}{2}[(t+s)^h+(t-s)^h]\right),\quad s\le t,
\end{aligned}\tag{3.15}$$

as $T\to\infty$, with $h=3-d/\alpha$.

The possible values of h are $h\in(1,2)$, and $h=3/2$ corresponds to $d=3$ and Brownian particle motion ($\alpha=2$).

The temporal part of (3.15) has the form of the covariance $C_h(s,t)$ of sub-fBm in (1.2) with $h>1$.

It is worthwhile to compare the previous results for the branching system with those for the system in equilibrium. The Poisson random measure is an equilibrium state for the system without branching but not for the branching system. The branching system has an equilibrium state for $d>\alpha$, which results from the interplay between the extinction of single families due to the critical branching and the replacement by families coming from far out by the transience of the motion (Gorostiza and Wakolbinger, 1991). We look now at what happens with the occupation time fluctuations of the branching system when the equilibrium state is taken as initial condition.

The covariance of the process N started from equilibrium (which can be derived from Dawson et al, 2001, or Gorostiza and Rodrigues, 1999) is given by

$$\mathrm{Cov}(\langle N_u,\varphi\rangle,\langle N_v,\psi\rangle)=\left\langle\lambda,\varphi\mathcal{T}_{v-u}\left(\psi+\frac{V}{2}G\psi\right)\right\rangle,\;u\le v,\tag{3.16}$$

and we note that it has the same form as in the non-branching case with ψ replaced by $\psi+\frac{V}{2}G\psi$ (see (3.3)). Then we have that for $\alpha<d<2\alpha$ and $F_T=T^{(3-d/\alpha)/2}$,

$$\mathrm{Cov}(\langle X_T(s),\varphi\rangle,\langle X_T(t),\psi\rangle)\to\langle\lambda,\varphi\rangle\langle\lambda,\psi\rangle\frac{V\Gamma(2-h)}{2^d\pi^{d/2}\alpha\Gamma(d/2)h(h-1)}(t^h+s^h-(t-s)^h),\;s\le t,\tag{3.17}$$

as $T\to\infty$, with $h=3-d/\alpha$.

We see from (3.8) and (3.17) that the system without branching (which is in equilibrium) for $d<\alpha$ and the branching system in equilibrium for $\alpha<d<2\alpha$ have the same temporal occupation time fluctuation limits.

We summarize the results on long-range dependence in the following table:

	Initial condition	Dimension	Norming	Temporal structure of limit
No branching	Poisson	$d<\alpha$	$T^{1-d/2\alpha}$	fBm
Branching	Poisson	$\alpha<d<2\alpha$	$T^{(3-d/\alpha)/2}$	sub-fBm
Branching	Equilibrium	$\alpha<d<2\alpha$	$T^{(3-d/\alpha)/2}$	fBm

The main observation is that, although for $d>\alpha$ the state of the branching system with Poisson initial condition approaches equilibrium as time flows (Gorostiza and Wakolbinger, 1994), in the occupation time fluctuations the memory of the initial condition is not forgotten but persists forever, and this results in sub-fBm as opposed to fBm. This difference exhibits explicitly the long memory of the occupation time process of the branching particle system for $\alpha < d < 2\alpha$.

Concerning the interpretation of our results for the particle systems, we stress that in this paper we relate the limit processes to the corresponding fluctuation processes of particle systems via their covariances only. Much stronger results on functional convergence are obtained in Bojdecki et al (2004).

Remarks

(a) Occupation time results for $d \geq \alpha$ in the non-branching case and for $d \geq 2\alpha$ in the branching case are not relevant for the present paper because they do not lead to long-range dependence (the time structure of the limit covariances is Bm in these cases).

(b) Another system which also leads to sub-fBm is the following. Let

$$M_t = \frac{1}{\sqrt{2}}(N_t + N^-_{-t}), \quad t \geq 0,$$

where N_t is the empirical measure of the non-branching system at time t and N^-_{-t} is the empirical measure of the system at time $-t$ (the non-branching system is defined for all $t \in (-\infty, \infty)$) but with the particles having negative charge -1. The occupation time fluctuations of the process M_t for $d < \alpha$ yield sub-fBm with $h = 2 - d/\alpha$.

(c) Concerning the conditions on d and α related with long-range dependence, we note that for the α-stable process, $d < \alpha$ corresponds to "strict recurrence" (i.e., recurrence except at the border to transience, $d = \alpha$) and $\alpha < d < 2\alpha$ corresponds to "strict weak transience" (i.e., weak transience except at the border to strong transience, $d = 2\alpha$). See e.g. Sato (1999) for recurrence/transience of α-stable processes.

(d) The denominator $|z|^\alpha$ in formulas (3.4) and (3.5) is related to the condition $d > \alpha$ for the branching system. However, $z = 0$ is not a singularity. A change of variable can be made before the last step in (3.4) ($z = T^{-1/\alpha}(u+v-2r)^{-1/\alpha}y$), and this leads to a limit without requiring the condition $d > \alpha$, which coincides with (3.15) if $d > \alpha$. The reason for assuming $d > \alpha$ from the beginning is that this is the necessary and sufficient condition for persistence of the branching particle system (persistence/extinction dichotomy, Gorostiza and Wakolbinger, 1991).

Note

After this article was completed we learned about the paper of Dzhaparidze and van Zanten (2004). They also consider a process of the form (2.1) (dividing by 2 rather than $\sqrt{2}$), regarded as the "even" part of fBm. Their results are a series expansion of the process and an integral representation with respect to Bm.

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